Groupoid extensions

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based on joint work [IKRSW 20] with

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Introduction

The Mackey normal subgroup analysis (also called the Mackey machine) describes the representations of a group G in terms of a normal subgroup S and the quotient $H = G/S$.

A semidirect product of groups

$$G = S \rtimes H$$

such as the group of rigid motions or the Poincaré group is the simplest example.

cont'd

From the C*-algebraic perspective, it gives a description of $C^*(G)$ as a crossed product. In the simple case of a semidirect product, we easily have

$$C^*(S \rtimes H) = C^*(S) \rtimes H = C^*(H, C^*(S))$$

A semidirect product is a trivial extension. When the extension is not trivial, a twist appears:

Theorem ([Green 78])

$$C^*(G) = C^*(G, C^*(S), \tau_S).$$

where the right handside is a twisted crossed product.

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When S is abelian, we can go one step further, namely use the Fourier-Gelfand transform

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The last term is a groupoid C*-algebra, where the groupoid $\hat{S} \rtimes H$ has less isotropy than the initial group $S \rtimes H$. One may want to iterate the process. It is then necessary to extend the Mackey machine to a groupoid G rather than a group. My original motivation was the analysis of nilpotent group C*-algebras.

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Plan

1. **Theorem**
2. **Example**
3. **Groupoid C*-dynamical systems and Fell bundles**
4. **Proof of the theorem**
5. **Abelian Fell bundles**

Groupoid extensions and twists

Definition

A groupoid extension $S \rightarrowtail G \twoheadrightarrow H$ is a surjective homomorphism $\pi : G \to H$, where G and H are groupoids having the same unit space, $\pi^{(0)} : G^{(0)} \to H^{(0)}$ is the identity map and $S = \operatorname{Ker}\pi$ is its kernel, i.e. the set of elements $\gamma \in G$ such that $\dot\gamma := \pi(\gamma) \in H^{(0)}$.

Note that S is a group bundle over $G^{(0)}$ contained in the isotropy group bundle of G and that G acts on S by conjugation.

Definition

A central groupoid extension $G^{(0)} \times \mathbb{T} \rightarrowtail \Sigma \twoheadrightarrow G$, where $\mathbb{T}$ is the group of complex numbers of module 1, is called a twist. Then, we say that (G, Σ) is a twisted groupoid.

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Analytic assumptions

We shall consider groupoid extensions $S \rightarrowtail G \twoheadrightarrow H$ such that G and H are locally compact Hausdorff groupoids, $\pi : G \to H$ is open and continuous, H has a Haar system α and S has a Haar system β. Then G has a Haar system λ given by

$$\int f(\gamma) d\lambda^x(\gamma) = \int_G \int_S f(\gamma t) d\beta^{s(\gamma)}(t) d\alpha^x(\dot{\gamma})$$

We shall say that $S \rightarrowtail G \twoheadrightarrow H$ is a locally compact groupoid extension with Haar systems. Then there exists a continuous homomorphism $\Delta : G \to \mathbb{R}_+^*$, called the modular function of the extension, such that

$$\gamma \beta^{s(\gamma)} \gamma^{-1} = \Delta(\gamma) \beta^{r(\gamma)}$$

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Statement

Theorem (cf. [IKRSW 20])

Let (G, Σ) be a twisted locally compact groupoid with Haar system. Let $S \rightarrowtail G \twoheadrightarrow H$ be a groupoid extension such that

1. *S is abelian: i.e. for all $x \in G^{(0)}$, S_x is an abelian group,*
2. *S possesses a Haar system β,*
3. *the restriction $\Sigma_{|S}$ of the twist to S is trivial.*

Then $C^*(G, \Sigma) = C^*(\hat{S}^\Sigma \rtimes H, \Sigma')$ where $\hat{S}^\Sigma$ is the spectrum of $C^*(S, \Sigma_{|S})$ and Σ' is a twist over $\hat{S}^\Sigma \rtimes H$ obtained by a pushout construction.

The canonical commutation relation

Let (V, ω) be a symplectic finite dimensional real vector space. Then, $\sigma = e^{i\omega/2}$ is a $\mathbb{T}$-valued 2-cocycle on $(V, +)$.

A CCR is a unitary σ-representation or equivalently a representation of the twisted group C*-algebra $C^*(V, \sigma)$.

We choose a direct decomposition $V = L \oplus L'$, where L and L' are Lagrangian subspaces. Applying the theorem to the extension

$$L \rightarrowtail V \twoheadrightarrow L'$$

gives that $C^*(V, \sigma) = C^*(L' \rtimes L')$ where L' acts on itself by translation. Here, the twist Σ' is trivial. Since the second term is the elementary C*-algebra $\mathcal{K}(L^2(L'))$, one retrieves the well-known von Neumann uniqueness theorem.

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Fell bundles over groupoids

It is useful at this stage to introduce the notion of Fell bundle over a groupoid, due to Yamagami and developed by Kumjian.

Definition ([Kumjian 98])

A Fell bundle over a groupoid is a pair $(H, \mathcal{B})$ where H is a locally compact groupoid and $\mathcal{B}$ is an upper semi-continuous bundle of Banach spaces over H endowed with a continuous multiplication $\mathcal{B} * \mathcal{B} \to \mathcal{B}$ and a continuous involution $\mathcal{B} \to \mathcal{B}$ satisfying the C*-algebra axioms whenever they make sense.

This definition implies that the fibers B_x over $x \in H^{(0)}$ become C*-algebras, the fibers B_γ over $\gamma \in H$ become $(B_{r(\gamma)}, B_{s(\gamma)})$ C*-bimodules and $\gamma \mapsto B_\gamma$ is functorial. One says that the Fell bundle is saturated if the B_γ's are equivalence C*-bimodules.

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Sectional algebras

The sectional C*-algebra of a Fell bundle $(H, \mathcal{B})$, where H has a Haar system α, is constructed from the $*$-algebra $C_c(H, \mathcal{B})$ whose elements are continuous compactly supported sections $F : H \to \mathcal{B}$. The product and the involution are respectively given by

$$F * G(\gamma) = \int F(\eta)G(\eta^{-1}\gamma)d\alpha^{r(\gamma)}(\eta)$$

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$$F^*(\gamma) = F(\gamma^{-1})^*$$

The C*-algebra $C^*(H, \mathcal{B})$ is obtained as the C*-completion for the full C*-norm.

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Groupoid C*-dynamical system

Definition (R 1987)

A groupoid C*-dynamical system is a triple $(G, \Sigma, \mathcal{A})$, where $S \rightarrowtail \Sigma \twoheadrightarrow G$ is a groupoid extension and $\mathcal{A}$ is an upper semi-continuous bundle of C*-algebras over $G^{(0)}$ endowed with a continuous action $(\sigma, a) \in \Sigma * \mathcal{A} \mapsto \sigma.a \in \mathcal{A}$ such that S is unitarily implemented in the multiplier algebra bundle $M(\mathcal{A})$. This means the existence of a bundle homomorphism $\chi : S \to UM(\mathcal{A})$ such that

1. the map $(s, a) \in S * \mathcal{A} \mapsto \chi(s)a \in \mathcal{A}$ is continuous,
2. $s.a = \chi(s)a\chi(s)^{-1}$,
3. $\chi(\sigma s \sigma^{-1}) = \sigma.\chi(s)$.

From dynamical systems to Fell bundles

Given a groupoid C*-dynamical system $(G, \Sigma, \mathcal{A})$ as above, we construct a Fell bundle $\mathcal{B}$ over G as follows. We form $\mathcal{A} * \Sigma = \{(a, \sigma) : a \in A_{r(\sigma)}\}$. We let S act on it by $s(a, \sigma) = (a\chi(s^{-1}), s\sigma)$ and consider the quotient $\mathcal{B} = (\mathcal{A} * \Sigma)/S$. The image of (a, σ) in $\mathcal{B}$ is denoted by $[a, \sigma]$. The bundle map $p : \mathcal{B} \to G$ sends $[a, \sigma]$ to $\pi(\sigma)$. A choice of σ in $\pi^{-1}(\gamma)$ gives a Banach space isomorphism $[a, \sigma] \mapsto a$ from B_γ to $A_{r(\gamma)}$. The multiplication in $\mathcal{B}$ is given by

$$[a, \sigma][b, \tau] = [a(\sigma.b), \sigma\tau]$$

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Crossed products

We recall the construction of the crossed products. Given a groupoid C*-dynamical system $(G, \Sigma, \mathcal{A})$ where G has a Haar system α, one first forms the $*$-algebra $C_c(G, \Sigma, \mathcal{A})$. Its elements are continuous functions $f : \Sigma \to \mathcal{A}$ such that

1. $f(\sigma)$ belongs to $A_{r(\sigma)}$ for all $\sigma \in \Sigma$;
2. $f(s.\sigma) = f(\sigma)\chi(s^{-1})$ for all $(s, \sigma) \in S * \Sigma$;
3. f has compact support modulo S.

The product and the involution are respectively given by

$$f * g(\sigma) = \int f(\tau)[\tau.g(\tau^{-1}.\sigma)] d\alpha^{r(\sigma)}(\dot{\tau})$$

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The C*-algebra $C^*(G, \Sigma, \mathcal{A})$ is the completion of $C_c(G, \Sigma, \mathcal{A})$ for the full C*-norm.

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The C*-algebras are the same

Not surprisingly, starting with a groupoid C*-dynamical system $(G, \Sigma, \mathcal{A})$, the associated C*-algebras are the same:

Proposition

Let $(G, \Sigma, \mathcal{A})$ be a groupoid C-dynamical system and let $(G, \mathcal{B})$ be the associated Fell bundle. Then the C*-algebras $C^*(G, \Sigma, \mathcal{A})$ and $C^*(G, \mathcal{B})$ are canonically isomorphic.*

The isomorphism simply associates to $f \in C_c(G, \Sigma, \mathcal{A})$ the function $F \in C_c(G, \mathcal{B})$ such that $F(\dot{\sigma}) = [f(\sigma), \sigma]$ for all $\sigma \in \Sigma$.

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Outline

We can now give the outline of the proof. Let (G, Σ) be a twisted groupoid and let $S \rightarrowtail G \twoheadrightarrow H$ be a groupoid extension as in the statement of the theorem.

1. Write the twisted groupoid C*-algebra $C^*(G, \Sigma)$ as the sectional algebra $C^*(H, \mathcal{B})$ of a Fell bundle over H (it is a particular case of [BM 16, Theorem 6.2]).
2. Observe that, under our assumptions, $\mathcal{B}$ is an abelian Fell bundle.
3. A structure theorem for abelian Fell bundles over groupoids (cf. [DKR 08, Theorem 5.6]) gives the equality $C^*(H, \mathcal{B}) = C^*(Z \rtimes H, \Sigma')$, where Z is the spectrum of the sectional algebra of the restriction of $\mathcal{B}$ to $H^{(0)}$ and Σ' is a twist over $Z \rtimes H$. We are (almost) done.

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The tautological Fell bundle

Let (G, Σ) be a twisted groupoid and let $\pi : G \twoheadrightarrow H$ be an extension with kernel S. Given $h \in H$, we set $G(h) = \pi^{-1}(h)$ and $L(h) = L_{|G(h)}$ where L is the line bundle associated with the circle bundle Σ. We consider the pre-Banach bundle over H having $C_c(G(h), L(h))$ as fibres and $C_c(G, L)$ as fundamental family of continuous sections. Let $(h, h') \in H^{(2)}$. Given $f \in C_c(G(h), L(h))$ and $g \in C_c(G(h'), L(h'))$, we define $f * g \in C_c(G(hh'), L(hh'))$ and $f^* \in C_c(G(h^{-1}), L(h^{-1}))$ by

$$f * g(\tau\tau') = \int f(\tau t)g(t^{-1}\tau')d\beta^{s(\tau)}(t) \quad \text{and} \quad f^*(\gamma^{-1}) = \overline{f(\gamma)}$$

where $\tau \in G(h), \tau' \in G(h')$ and $\gamma \in G(h)$.

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Note that for $h = x \in H^{(0)}$, $G(x)$ is the group S_x. Hence we endow the $*$-algebra $C_c(G(x), L(x)) \simeq C_c(S_x)$ with the group C*-norm and we deduce from that a norm on each $C_c(G(h), L(h))$. By completion, we obtain a Fell bundle $\mathcal{B}$ over H which we call the tautological Fell bundle.

Proposition ([BM 16])

$C^*(G, \Sigma)$ is canonically isomorphic to the sectional algebra $C^*(H, \mathcal{B})$.

The proof is straightforward: to $f \in C_c(G, L)$ associate the field $h \mapsto f_{|G(h)}$ which is an element of $C_c(H, \mathcal{B})$. This is a $*$-algebra isomorphism which extends to an isomorphism of the C*-algebras.

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Proposition ([BM 16])

$C^*(G, \Sigma)$ is canonically isomorphic to the sectional algebra $C^*(H, \mathcal{B})$.

The proof is straightforward: to $f \in C_c(G, L)$ associate the field $h \mapsto f_{|G(h)}$ which is an element of $C_c(H, \mathcal{B})$. This is a $*$-algebra isomorphism which extends to an isomorphism of the C*-algebras.

The tautological groupoid dynamical system

The tautological Fell bundle $\mathcal{B}$ is in fact the Fell bundle of the dynamical system $(H, G, \mathcal{C}^*(S, \Sigma_{|S}))$ which we describe now.
The upper semi-continuous bundle of C*-algebras $\mathcal{C}^*(S, \Sigma_{|S})$ with fibers $C^*(S_x, \Sigma_x)$ is endowed with a continuous action of G, defined for $\sigma \in \Sigma$, $h \in C_c(S_{s(\sigma)}, \Sigma_{s(\sigma)})$ and $t \in S_{r(\sigma)}$ by

$$(\dot\sigma.h)(t) = \Delta(\dot\sigma)\sigma h(\dot\sigma^{-1}t\dot\sigma)\sigma^{-1}$$

Its restriction to S is unitarily implemented by $\chi : S \to UM(\mathcal{C}^*(S, \Sigma_{|S}))$ where $\chi(t)$ is the canonical unitary of $C^*(S_{s(t)}, \Sigma_{s(t)})$, which is

$$(\chi(t)h)(u) = \Delta^{1/2}(t)h(t^{-1}u)$$

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Construction of abelian Fell bundles

Let Σ be a twist over a semi-direct product $Z \rtimes G$, where G is a groupoid and Z a right G-space. We denote by $\mathcal{L}$ the associated Fell line bundle. Then, for each $\gamma \in G$, we obtain by restriction a line bundle $\mathcal{L}_\gamma$ over $Z_{r(\gamma)}$ and consider the space of continuous sections vanishing at infinity $B_\gamma := C_0(Z_{r(\gamma)}, \mathcal{L}_\gamma)$ endowed with the sup-norm. We turn $\mathcal{B} := \bigsqcup_{\gamma\in G} B_\gamma$ into a Banach bundle with $C_c(Z \rtimes G, \mathcal{L})$ as fundamental space of continuous sections, where we identify $f \in C_c(Z \rtimes G, \mathcal{L})$ with the section $\gamma \to f_\gamma$, where $f_\gamma \in C_c(Z_{r(\gamma)}, \mathcal{L}_\gamma)$ is defined by $f_\gamma(z) = f(z, \gamma)$.

Construction (cont'd)

Given $(\gamma,\gamma')\in G^{(2)}$, we define the product

$$B_\gamma\otimes B_{\gamma'}\to B_{\gamma\gamma'}: b\otimes b'\mapsto b(\gamma b')$$

where, for $z\in Z_{r(\gamma)}$, $(b(\gamma b'))(z)=b(z)b'(z\gamma)$. For $\gamma\in G$, we define the involution

$$B_\gamma\to B_{\gamma^{-1}}: b\to b^*$$

where, for $z\in Z_{s(\gamma)}$, $b^*(z)=(b(z\gamma^{-1}))^*$.

Proposition

Let Σ a twist over the semi-direct product $Z\rtimes G$, where (G,α) is a locally compact groupoid with Haar system and Z a right locally compact G-space Z. Construct $\mathcal{B}$ as above. Then

1. *$\mathcal{B}$ is a saturated abelian Fell bundle over G;*
2. *$C^*(G,\mathcal{B})$ is isomorphic to $C^*(Z\rtimes G,\Sigma)$.*

The structure theorem for abelian Fell bundles

In fact, every saturated abelian Fell bundle can be obtained by this construction.

Theorem ([DKR 08])

Let $\mathcal{B}$ be a saturated abelian Fell bundle over a locally compact groupoid G. Then there exists a right locally compact G-space Z and a twist Σ over the semi-direct product $Z \rtimes G$ such that $\mathcal{B}$ is isomorphic to the Fell bundle obtained from $(Z \rtimes G, \Sigma')$. The pair (Z, Σ') is unique up to isomorphism.

Proof

The restriction of $\mathcal{B}$ to $G^{(0)}$ is an abelian C*-bundle $\mathcal{B}^{(0)}$ over $G^{(0)}$. We define Z as the spectrum of its sectional algebra $C_0(G^{(0)}, \mathcal{B}^{(0)})$. It carries a (right) G-action. We denote by $s : G \to G^{(0)}$ the projection map and $Z_x = s^{-1}(x)$ its fibre. For each $\gamma \in G$, $(B_{r(\gamma)}, B_\gamma, B_{s(\gamma)})$ is a Morita equivalence. The structure of a Morita equivalence of abelian C*-algebras is well known: it is given by a twisted homeomorphism, i.e. a homeomorphism $h_\gamma : Z_{s(\gamma)} \to Z_{r(\gamma)}$ together with a line bundle $\mathcal{L}_\gamma$ over $Z_{r(\gamma)}$. From this one can construct a Fell line bundle $\mathcal{L}$ over $Z \rtimes G$, whose unitary bundle is the desired twist Σ'.

Case of a groupoid dynamical system

Assume that $\mathcal{B}$ is the Fell bundle of a groupoid dynamical system $(G, \Sigma, \mathcal{A})$, where $\mathcal{A}$ is an abelian C*-bundle. Note that we do not assume that the kernel S of the extension $S \rightarrowtail \Sigma \twoheadrightarrow G$ is abelian. The sectional C*-algebra $A = C_0(G^{(0)}, \mathcal{A})$ is abelian, hence isomorphic to $C_0(Z)$, where Z is the spectrum of A. The action of Σ on $\mathcal{A}$ turns Z into a right G-space. The homomorphism $\chi : S \to UM(\mathcal{A})$ which implements the restriction of the action to S gives a map

$$\underline{\chi} : Z * S \to \mathbb{T} \qquad (z, t) \mapsto (\chi(t))(z)$$

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The push out twist

The following pushout diagram gives the twist Σ' of the structure theorem.

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\begin{array}{ccccc}
Z * S & \longrightarrow & Z \rtimes \Sigma & \longrightarrow & Z \rtimes G \\
{\scriptstyle \pi_1 \times \underline{\chi}} \big\downarrow & & \big\downarrow & & \big\| \\
Z \times \mathbb{T} & \longrightarrow & \Sigma' & \longrightarrow & Z \rtimes G
\end{array}
$$

Theorem

If $\mathcal{A}$ is abelian with $A = C_0(Z)$, then the twisted crossed product $C^*(G, \Sigma, \mathcal{A})$ is isomorphic to $C^*(Z \rtimes G, \Sigma')$.

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conclusion

- A. Buss and R. Meyer: *Iterated crossed products for groupoid fibrations*, arXiv 1604:02105.
- V. Deaconu, A. Kumjian and B. Ramazan: *Fell bundles associated to groupoid morphisms*, Math. Scand. 102 (2008) no 2. 305-319.
- P. Green: *The local structure of twisted covariance algebras*, Acta Mat. **140** (1978).
- M. Ionescu, A. Kumjian, J. Renault, A. Sims and D. Williams: *A Mackey normal subgroup analysis for groupoids*, arXiv 2001:01312.
- A. Kumjian: *Fell bundles over groupoids*, Proc. Amer. Math. Soc. **126, 4** (1998).

The End

THANK YOU!