KMS states and groupoid C*-algebras

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Plan

1. **KMS states**
2. **KMS measures**
3. **Nevshveyev's theorem**
4. **Examples**

Elementary quantum statistical mechanics

The algebra of observables is the full matrix algebra $M_n(\mathbb{C})$. Time evolution is given by a one-parameter group

$$\sigma_t(A) = e^{itH} A e^{-itH}$$

where $H \in M_n^{s.a.}(\mathbb{C})$ is the hamiltonian. One defines the entropy of the state $\varphi = Tr(\ .\ \Phi)$, where Φ is the density matrix, by
$S(\varphi) = -Tr(\Phi \log \Phi)$ and its free energy by
$F(\varphi) = S(\varphi) - \beta\varphi(H)$, where β is the inverse temperature.
The equilibrium state of the system, at fixed β and H, maximises the free energy. It is given by the following Gibbs Ansatz.

Gibbs state

Proposition

Let $H \in M_n^{s.a.}(\mathbb{C})$ et $\beta \in \mathbb{R}$.

- $F(\varphi) \leq Tr(e^{-\beta H})$
- *equality holds iff* $\Phi = e^{-\beta H}/Tr(e^{-\beta H})$.

This justifies the following definition.

Definition

The state with density matrix $\Phi = e^{-\beta H}/Tr(e^{-\beta H})$ is called the Gibbs state (for the hamiltonian H and at inverse temperature β).

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Let $H \in M_n^{s.a.}(\mathbb{C})$ *et* $\beta \in \mathbb{R}$.

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Infinite systems

In the C*-algebraic formalism of quantum theory, the algebra of observables is an arbitrary C*-algebra A. The above formula does not usually make sense, even when $A = \mathcal{K}(\mathcal{H})$ is the algebra of compact operators on an infinite dimensional Hilbert space $\mathcal{H}$. Indeed it requires $e^{-\beta H}$ to be trace class, which is not always satisfied. A possible way is the thermodynamical limit: one considers larger and larger finite subsystems. We are going to describe another way.

KMS states

Kubo, Martin and Schwinger have discovered a direct relation between the Gibbs state and the one-parameter group σ_t.

Definition

Let A be a C*-algebra, σ_t a strongly continuous one-parameter group of automorphisms of A and $\beta \in \mathbb{R}$. One says that a state φ of A is KMS$_\beta$ for σ if it is invariant under σ_t and for all $a, b \in A$, there exists a function F bounded and continuous on the strip $0 \leq Imz \leq \beta$ and analytic on $0 < Imz < \beta$ such that:

- $F(t) = \varphi(a\sigma_t(b))$ for all $t \in \mathbb{R}$;
- $F(t + i\beta) = \varphi(\sigma_t(b)a)$ for all $t \in \mathbb{R}$.

A state φ is called tracial if $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$. KMS states should be seen as generalizations of tracial states.

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KMS states and Gibbs states

In the elementary case considered earlier, the KMS states are exactly the Gibbs states:

Proposition

Let $A = \mathcal{K}(\mathcal{H})$ be the algebra of compact operators on a Hilbert space $\mathcal{H}$, $\beta \in \mathbb{R}$ and H a self-adjoint operator such that $e^{-\beta H}$ is trace class. Then the Gibbs state with density matrix $\Phi = e^{-\beta H}/Tr(e^{-\beta H})$ is the unique KMS_β-state for the one-parameter group generated by H.

Properties of KMS states

Here A is a separable C*-algebra and $\sigma = (\sigma_t)$ is a strongly continuous one-parameter group of automorphisms of A.

- The σ_t invariance is implied by KMS$_\beta$ for $\beta \neq 0$.
- For a given β, the set Σ_β of KMS$_\beta$-states is a Choquet simplex of A^*: i.e. it is a $*$-weakly closed convex subset of A^* and every KMS$_\beta$-state is the barycenter of a unique probability measure supported on the extremal KMS$_\beta$-states.
- The extremal KMS$_\beta$-states are factorial.

Problem. Determine all the KMS-states of a given dynamical system (A, σ). The discontinuities of the map $\beta \mapsto \Sigma_\beta$ are interpreted as phase transitions.

Definition of Cuntz algebras

Here is a basic example.

Definition

The Cuntz algebra O_n, where $n \in \mathbb{N}$, is the C*-algebra generated by n isometries $S_1, \ldots, S_n$ of a Hilbert space $\mathcal{H}$ whose ranges give an orhogonal decomposition of $\mathcal{H}$.

Thus we have the Cuntz relations

- $S_i^* S_j = \delta_{i,j} \mathrm{I}$
- $\sum_{i=1}^n S_i S_i^* = \mathrm{I}$.

One says "the Cuntz algebra O_n" because it is unique up to isomorphism.

Gauge group

Let $z = e^{it}$ be a complex number of modulus one. Then $zS_1, \ldots, zS_n$ satisfy the Cuntz relations and generate the same C*-algebra as O_n. Therefore, there exists a unique automorphism σ_t of O_n such that $\sigma_t(S_j) = e^{it} S_j$ for all $j = 1, \ldots, n$. This defines a strongly continuous automorphism group of O_n.

Definition

This one-parameter automorphism group $\sigma = (\sigma_t)$ is called the gauge group of the Cuntz algebra O_n.

As the Cuntz algebra contains non unitary isometries, it does not have tracial states. However it possesses a KMS state for the gauge group.

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The KMS state for the gauge group

Theorem (Olesen-Pedersen, Elliott, Evans)

The gauge group of the Cuntz algebra O_n has a unique KMS state. It occurs at the inverse temperature $\beta = \log n$.

Proof. A state φ is completely determined on the elements of the form $a = S_{i_1} \dots S_{i_k} S_{j_l}^* \dots S_{j_1}^*$. The invariance under σ gives $\varphi(a) = 0$ si $k \neq l$. Iterating the KMS condition, we obtain:

$$\varphi(a) = \delta_{i_1,j_1} \dots \delta_{i_k,j_k} e^{-k\beta}$$

The condition $\varphi(1) = 1$ and the second Cuntz relation give $1 = ne^{-\beta}$. Provided it exists, these relations determine uniquely the KMS state. One checks that these relations do determine a state.

Elementary QSM revisited

We choose an orthonormal basis which diagonalizes the hamiltonian H. We let $(h_1, \ldots, h_n)$ be its diagonal entries. Then the Gibbs state at inverse temperature β is given by the diagonal density matrix $\Phi = (\rho_1, \ldots, \rho_n)$ where the weights ρ_i are completely determined by

1. $\rho_i > 0$,
2. $\sum_{i=1}^n \rho_i = 1$ and
3. $\rho_i / \rho_j = e^{-\beta(h_i - h_j)}$.

We can give a sense to these conditions in a much wider framework.

Gibbs measure

Let μ be the probability measure on $\{1,\ldots,n\}$ defined by the weights ρ_i. Then above condition 3 can be expressed as the equation

$$D_\mu = e^{-\beta c}$$

where

- $c(i,j) = h_i - h_j$
- D_μ is the Radon-Nikodym derivative $d(r^*\mu)/d(s^*\mu)$, r, s are respectively the first and the second projections of $\{1,\ldots,n\} \times \{1,\ldots,n\}$ onto $\{1,\ldots,n\}$ and $r^*\mu, s^*\mu$ are the measures on $\{1,\ldots,n\} \times \{1,\ldots,n\}$ obtained by summing respectively the rows and the columns and integrating with respect to μ.

Groupoids

Under this form, this example can be generalized to arbitrary locally compact groupoids with Haar systems. Here are my notations:

range and source maps: $G \to G^{(0)} \times G^{(0)} : \gamma \mapsto (r(\gamma), s(\gamma))$

inverse map: $G \to G : \gamma \mapsto \gamma^{-1}$

inclusion map $i : G^{(0)} \to G : x \mapsto x$

product map $G^{(2)} \to G : (\gamma, \gamma') \mapsto \gamma\gamma'$.

$G^x = r^{-1}(x)$, $G_y = s^{-1}(y)$, $G_x^x = G(x) = G^x \cap G_x$.

Haar systems

We assume from now on that G is endowed with a locally compact topology compatible with its algebraic structure. When G is a group, it has a Haar measure, i.e. a left invariant measure. When G is a groupoid, we assume the existence of a Haar system:

Definition

A Haar system on a locally compact groupoid G is a family $\lambda = (\lambda^x)_{x \in G^{(0)}}$, where

- λ^x is a measure on $G^x = r^{-1}(x)$,
- for all $f \in C_c(G)$, the map $x \mapsto \int f d\lambda^x$ is continuous and
- for all $\gamma \in G$, $\gamma\lambda^{s(\gamma)} = \lambda^{r(\gamma)}$.

When G is étale, i.e. r is a local homeomorphism, the counting measures on the fibres G^x form a Haar system.

Quasi-invariant measures

Definition

Let (G,λ) be a locally compact groupoid with Haar system. A measure μ on $G^{(0)}$ is called quasi-invariant if the measures $\mu\circ\lambda$ and its inverse $(\mu\circ\lambda)^{-1}$ are equivalent. We denote by

$$\boxed{D_\mu = \frac{d(\mu\circ\lambda)}{d(\mu\circ\lambda)^{-1}}}$$

the Radon-Nikodym derivative.

This agrees with the usual definition when G is the groupoid of the action of a locally compact group on a space.

Cocycles

Definition

Let G be a groupoid and A a group. An A-valued cocycle on G is a groupoid morphism $c : G \to A$.

Proposition

Let μ be a quasi-invariant measure. Then the Radon-Nikodym derivative D_μ is a $\mathbb{R}_+^*$-valued cocycle.

This is the chain rule, just as in the case of the groupoid of the action of a locally compact group on a space.

KMS measures

We can generalize the elementary example as follows.

Definition

Let (G, λ) be a locally compact groupoid with Haar system, let $c : G \to \mathbb{R}$ be a continuous cocycle and let $\beta \in \mathbb{R}$. One says that a measure μ on $G^{(0)}$ is KMS_β for the cocycle c if it is quasi-invariant and satisfies $\boxed{D_\mu = e^{-\beta c}}$.

The properties of the KMS_β-probability measures are similar to those of the KMS_β-states. For example, if $G^{(0)}$ is compact, they form a Choquet simplex. The extremal elements are exactly the ergodic ones. The above definition is essentially the same as the definition of Gibbs measures given by Capocaccia in the framework of statistical mechanics in 1976. It also agrees with the Dobrushin-Lanford-Ruelle definition.

The groupoid C*-algebra

Let (G, λ) be a locally compact groupoid with a Haar system. One constructs a C*-algebra exactly like a matrix algebra. One first consider the $*$-algebra $C_c(G)$ of continuous and compactly supported functions on G. The product and the involution are given by

$$f * g(\gamma) = \int f(\gamma\gamma')g(\gamma'^{-1})d\lambda^{r(\gamma)}(\gamma'); \qquad f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

Every representation L of this $*$-algebra as bounded operators on a Hilbert space gives a semi-norm $\|L(f)\|$. One gets the full norm by taking the supremum of these semi-norms over all representations which are continuous in the inductive limit topology. Its completion is denoted by $C^*(G)$.

Diagonal automorphism groups

Proposition

Let $c : G \to \mathbb{R}$ *be a continuous cocycle. Then the formula*

$$\boxed{\sigma_t^c(f)(\gamma) = e^{itc(\gamma)} f(\gamma)}, \qquad f \in C_c(G)$$

defines a strongly continuous one-parameter automorphism group σ^c *of* $C^*(G)$.

Definition

A strongly continuous one-parameter automorphism group σ of a C*-algebra A is called diagonal if there exists a locally compact groupoid with Haar system (G, λ) and a continuous cocycle $c : G \to \mathbb{R}$ such that (A, σ) is isomorphic to $(C^*(G), \sigma^c)$.

Disintegration of KMS states

Theorem (R80, Kumjian-R06, Neshveyev13)

Let G, c and β as above. Assume that G is étale and that $G^{(0)}$ is compact. Then,

1. *Given a KMS_β-probability measure μ on $G^{(0)}$ and a measurable family of states φ_x on the subgroups $G_x^x \cap c^{-1}(0)$ such that $\gamma\varphi_{s(\gamma)}\gamma^{-1} = \varphi_{r(\gamma)}$ for all $\gamma \in G$, the formula*

$$\varphi(f) = \int \varphi_x(f)d\mu(x), \qquad f \in C_c(G)$$

   defines a KMS_β-state for σ^c.

2. *All KMS_β-states for σ^c have the above form.*

Comment

In his recent thesis, J. Christensen extends this theorem to the case when $G^{(0)}$ is no longer compact. He has then to consider KMS weights rather than KMS states.

I suspect that a version of this theorem holds for non-étale locally compact groupoids with Haar system.This requires to consider weights, which is always a delicate business.

The Renault-Deaconu groupoid

Given a topological space X, an open subset U and a local homeomorphism $T : U \to X$, we build the following semi-direct product groupoid:

$$G(X,T) = \{(x, m-n, y) : x, y \in X;\ m, n \in \mathbb{N} \text{ et } T^m x = T^n y\}$$

It is implicit in this definition that x [resp. y] belongs to the domain of T^m [resp. T^n]. If X is locally compact and Hausdorff, then $G(X,T)$ is a locally compact étale Hausdorff groupoid.

Two basic examples (here, $U = X$).

- The one-sided shift on $\prod_1^\infty \{0,1\}$.
- The map $z \mapsto z^2$ on the circle.

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Quasi-product cocycles

Given $\phi \in C(X, \mathbb{R})$, we define the cocycle $c : G(X, T) \to \mathbb{R}$ by

$$c_\phi(x, m-n, y) = \sum_{k=0}^{m} \phi(T^k x) - \sum_{l=0}^{n} \phi(T^l y)$$

Similarly, given $\psi \in C(X, \mathbb{R}_+^*)$, we define the cocycle $D_\psi : G(X, T) \to \mathbb{R}_+^*$ by

$$D_\psi(x, m-n, y) = \frac{\psi(x)\psi(Tx)\dots\psi(T^m x)}{\psi(y)\psi(Ty)\dots\psi(T^n y)}$$

and the transfer operator $L_\psi : C_c(U) \to C_c(X)$ by

$$L_\psi f(y) = \sum_{x \in T^{-1}(\{y\})} \psi(x) f(x).$$

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Conformal measures

In the thermodynamical formalism, our KMS measures are called conformal measures. They can be described as Perron-Frobenius eigenfunctions of the dual of the transfer operator.

Lemma

A Radon measure μ on X is quasi-invariant with R-N derivative D_ψ iff $\boxed{L_\psi^*\mu = \mu_{|U}}$.

Corollary

Given $\varphi \in C(X, \mathbb{R})$ and $\beta \in \mathbb{R}$, a Radon measure μ on X is KMS_β for the cocycle c_φ iff $L_{e^{-\beta\varphi}}^*\mu = \mu_{|U}$.

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Perron-Frobenius-Walters theorem

Here is a well-known result in the thermodynamical formalism for topological dynamical systems.

Theorem (Walters)

Assume X compact and $T : X \to X$ positively expansive (there is a neighborhood W of the diagonal $\Delta \subset X \times X$ such that for $(x, y) \notin \Delta$, there is n such that $(T^n x, T^n y) \notin W$) and exact (the equivalence relation $T^n x = T^n y$ for some n is minimal). Let $\psi \in C(X, \mathbb{R}_+^*)$. Then,

1. *The equation $L_\psi^* \mu = \lambda\mu$ where μ is a probability measure admits a unique positive solution λ;*
2. $\log \lambda = P(T, \log \psi)$ *where P denotes the pression;*
3. *if ψ satisfies Bowen's condition, the measure μ is unique.*

An existence and uniqueness result

This gives in our framework a result about existence and uniqueness of KMS-states.

Corollary (K-R06)

Assume that T is positively expansive and exact and that $\varphi \in C(X, \mathbb{R})$. Define $A = C^*(X, T)$ and σ as earlier. Then

1. *There exists a KMS_β-state for σ iff $P(T, -\beta\varphi) = 0$;*
2. *if φ has a constant sign and e^φ satisfies Bowen's condition, there exists one and only one KMS-state for σ.*

Our example of the Cuntz algebra is a particular case:

the Bernoulli shift

Example

Here $X = \{1, \ldots, n\}^{\mathbb{N}}$ and $T(x_0 x_1 \ldots) = x_1 x_2 \ldots$. Then, $C^*(X, T)$ is the Cuntz algebra O_n.
The function $\varphi \equiv 1$ defines the gauge group σ. Above condition 2 is satisfied. One retrieves the uniqueness of the KMS state. The equation $P(T, -\beta\varphi) = 0$ gives $\beta = h(T) = \log n$.

This example admits many generalizations. First, we may consider more general potentials. For example, if $\varphi(x) = \lambda_{x_0}$ where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, the equation $P(T, -\beta\varphi) = 0$ becomes

$$\sum_{i=1}^{n} e^{-\beta\lambda_i} = 1$$

It admits solutions iff $\lambda_1, \ldots, \lambda_n$ have same sign. Then, the solution is unique.

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Graph algebras

The same technique applies to subshifts of finite type. The associated C*-algebras are the Cuntz-Krieger algebras. When the transition matrix is primitive, the gauge group has a single KMS state occuring at inverse temperature $\beta = \log \lambda$, where λ is the Perron-Frobenius eigenvalue of the transition matrix.

Graph algebras are generalizations of Cuntz-Krieger algebras. They can also be presented as C*-algebras of a Renault-Deaconu groupoid. Their KMS states have been thoroughly investigated. The groupoid techniques do apply.

Exel-Laca algebras

Exel and Laca have defined in 1999 the Cuntz-Krieger algebra of an infinite matrix $M : I \times I \to \{0,1\}$ where the index set I is infinite countable. For convenience, we introduce the oriented graph (V, E) where the set of vertices is $V = I$ and the arrows are (i,j) where $M(i,j) = 1$. When M is not row-finite, the space X_∞ of infinite paths is not locally compact. Let us define $J(j)$ as the set of arrows (i,j) and $\mathcal{J}$ as the set of limit points of $J(j)$ as $j \to \infty$.

Definition

A terminal path is

- either an infinite path $i_0 i_1 i_2 \ldots$
- or a controlled finite path $(i_0 i_1 i_2 \ldots i_n; J)$ where $J \in \mathcal{J}$ and $i_n \in J$;
- or an empty path $(\emptyset; J)$ where $J \in \mathcal{J}$.

Exel-Laca algebra as a groupoid C*-algebra

Proposition (R 99)

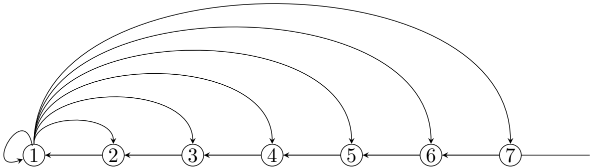
1. *the set of terminal paths* $X = X_\infty \sqcup X_{\mathrm{f}}$ *admits a natural locally compact topology;*
2. *the shift* $T : U \to X$*, where* $U = X \setminus \{(\emptyset; J), J \in \mathcal{J}\}$*, is a local homeomorphism.*

Theorem (R 99)

Exel-Laca algebra is the groupoid C-algebra* $C^*(G(X, T))$.

The renewal shift

Bissacot, Exel, Frausino, Raszeja have recently revisited the theory of conformal measures on countable Markov chains, using the framework of Exel-Laca algebras. Their pet example is the famous renewal shift.



Conformal measures on the renewal shift

Let us first determine the terminal path space of the renewal shift. Since $J(j) = \{0, j+1\}$, the only limit point is the set $\{0\}$. The controlled finite paths are the finite paths which end by 0. We have $X = X_\infty \sqcup X_{\mathrm{f}}$ and $U = X \setminus \{(\emptyset; \{0\})\}$.

Theorem (Bissacot, Exel, Frausino, Raszeja)

Consider a potential $\varphi : X \to \mathbb{R}$ such that $\varphi(i_0 i_1 i_2 \ldots) = f(i_0)$ where $f : I \to \mathbb{R}$ admits a strictly positive infimum $M > 0$. Then

1. *if $\beta > \log 2/M$, there exists a unique (c_φ, β)-KMS measure which vanishes on X_∞;*
2. *if $\beta < \log 2/M$, there exists no (c_φ, β)-KMS measure which vanish on X_∞.*

Bost-Connes dynamical system

It is an example of a C*-dynamical system (A, σ) coming from number theory which exhibits a phase transition. The C*-algebra A comes from the Hecke pair:

$$P_{\mathbb{Z}}^{+} := \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \subset \begin{pmatrix} 1 & \mathbb{Q} \\ 0 & \mathbb{Q}_{+}^{*} \end{pmatrix} := P_{\mathbb{Q}}^{+}$$

The subgroup $P_{\mathbb{Z}}^{+}$ is not normal but almost normal in the sense that the double cosets contain only a finite number of right (and left) cosets. The C*-algebra A is the regular C*-completion of the Hecke algebra, i.e. the convolution algebra of functions on the double cosets. The automorphism group σ arises from the number of right coset in a double coset.

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Theorem (Bost-Connes, 95)

Let (A, σ) be the Bost-Connes system.

1. *For all $0 < \beta \leq 1$, there exists one and only one KMS$_\beta$-state. It generates a III_1 factor. It is invariant under the action of Aut$(\mathbb{Q}/\mathbb{Z})$.*
2. *For all $1 < \beta \leq \infty$, extremal KMS$_\beta$-states are parametrized by the embeddings $\chi : \mathbb{Q}^{\mathrm{cycl}} \to \mathbb{C}$. They generate the I_∞-factor. The group Aut$(\mathbb{Q}/\mathbb{Z})$ acts freely and transitively on the set of extremal KMS$_\beta$-states.*
3. *The partition function of this system is the Riemann zeta function.*

About the proof

One can apply the groupoid technique and write the action as a diagonal action. Indeed $A = C^*(G)$ where G is the groupoiid

$$G = \{(x, m/n, y) \in \mathcal{R} \times \mathbb{Q}_+^* \times \mathcal{R} : mx = ny\}$$

with

- $m, n \in \mathbb{N}^*$;
- $\mathcal{R} = \prod \mathbb{Z}_p$;
- $\mathbb{Z}_p$ is the ring of p-adic integers;
- the product is over the set $\mathcal{P}$ of prime numbers;
- $\mathbb{N}^*$ is embedded into $\mathbb{Z}_p$ for each p, hence into $\mathcal{R}$ by the diagonal embedding.

As in the example of the gauge group of the Cuntz algebra, the automorphism group is diagonal and given by the cocycle $c : G \to \mathbb{R}$ defined by

$$c(x, m/n, y) = \log(m/n)$$

Since the assumptions of the theorem [KR] are satisfied, the problem is reduced to solving the equation $D_\mu = e^{-\beta c}$.

As an intermediate step, one studies

$$H = \{(x, m/n, y) \in \mathcal{N} \times \mathbb{Q}_+^* \times \mathcal{N} : mx = ny\}$$

with

- $m, n \in \mathbb{N}^*$;
- $\mathcal{N} = \prod_{\mathcal{P}} \overline{\mathbb{N}}$ is the space of generalized integers, given by $2^{n_2} 3^{n_3} \ldots$
- $\mathbb{N}^*$ is a subset of $\overline{\mathbb{N}}$, hence by diagonal embedding, a subset of $\mathcal{N}$.

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