Crossed products by automorphisms of $C(X, D)$

N. Christopher Phillips

University of Oregon

2 October 2020

Western Sydney, IPM joint workshop on operator algebras.

30 Sept.–2 Oct. 2020

Joint work with Dawn Archey and Julian Buck.

A rough outline

This is a corrected version.

Outline:

- Introduction.
- An extended outline.
- Some definitions.
- Why we need conditions on the automorphism.
- Some examples.

Introduction

There has been extensive work on crossed products $C^*(\mathbb{Z}, X, h)$ for minimal homeomorphisms h, and on crossed products $C^*(\mathbb{Z}, A, \alpha)$ when A is simple and α is "sufficiently outer" that $C^*(\mathbb{Z}, A, \alpha)$ is simple. Here we consider the most tractable intermediate situation, in which A is neither commutative nor simple: $A = C(X, D)$ with X compact metrizable and D simple, separable, and unital, but still with hypotheses to ensure that $C^*(\mathbb{Z}, A, \alpha)$ is simple. To avoid triviality (reduction to earlier cases), we assume throughout X is infinite.

Let $\alpha \in \mathrm{Aut}(A)$. Since $\mathrm{Prim}(A) \cong X$, α induces a homeomorphism $h \colon X \to X$, and there is a function $x \mapsto \alpha_x \in \mathrm{Aut}(D)$ such that $\alpha(a)(x) = \alpha_x(a(h^{-1}(x)))$ for all $a \in C(X, D)$ and $x \in X$. One can check that the function $x \mapsto \alpha_x$ comes from an automorphism if and only if it is continuous in the topology of pointwise convergence in the norm of A, that is, $x \mapsto \alpha_x(a)$ is continuous for all $a \in D$.

Introduction (continued)

$A = C(X, D)$ with X compact metrizable and infinite, and D simple, separable, and unital.

$h \colon X \to X$ is a homeomorphism.

$x \mapsto \alpha_x \in \mathrm{Aut}(D)$ is continuous in the topology of pointwise norm convergence, and $\alpha \in \mathrm{Aut}(A)$ is given by $\alpha(a)(x) = \alpha_x(a(h^{-1}(x)))$.

We say that α *lies over h*. We can specify α by giving X, h, D, and the map $x \mapsto \alpha_x$.

Under these hypotheses (including X being infinite), one can check (using Archbold and Spielberg) that $C^*(\mathbb{Z}, A, \alpha)$ is simple if and only if h is minimal.

A very basic example: $\gamma \in \mathrm{Aut}(D)$, $h \colon X \to X$ is minimal, and the action on $C(X, D) = C(X) \otimes D$ is the tensor product of $f \mapsto f \circ h^{-1}$ and γ. Even here, little seems to be known about the crossed product.

Introduction (continued)

$A = C(X, D)$ with X compact metrizable and infinite, and D simple, separable, and unital, $h \colon X \to X$ is a minimal homeomorphism, and $x \mapsto \alpha_x \in \mathrm{Aut}(D)$ is continuous.

We want to find conditions on D and h which ensure that $C^*(\mathbb{Z}, A, \alpha)$ has good structural properties, such as:

- Stable rank one.
- Real rank zero.
- Strict comparison of positive elements.
- Z-stability.
- Tracial Z-stability.
- Pure infiniteness.

These properties might come from D, from h, or from some combination.

As will be seen, for now we often need to restrict to special choices of X. But there are still many examples which are not accessible with previously known techniques.

Some general families of examples

We choose X, $h\colon X \to X$, a simple separable unital C*-algebra D, and a continuous map $x \mapsto \alpha_x$ from X to $\mathrm{Aut}(D)$. Then $\alpha \in \mathrm{Aut}(C(X, D))$ is $\alpha(a)(x) = \alpha_x(a(h^{-1}(x)))$ for $a \in C(X, D)$ and $x \in X$.

Example

Let $h\colon X \to X$ be any minimal homeomorphism, let $\beta \in \mathrm{Aut}(C(X))$ be $\beta(f) = f \circ h^{-1}$, let D and $\gamma \in \mathrm{Aut}(D)$ be arbitrary, and set $\alpha = \beta \otimes \gamma \in \mathrm{Aut}(C(X) \otimes D)$.

This comes from $\alpha_x = \gamma$ for all $x \in X$.

Example

Index the generators s_j of $\mathcal{O}_\infty$ by $j \in \mathbb{Z}$. Take $X = (S^1)^{\mathbb{Z}}$, and for $\zeta = (\zeta_k)_{k \in \mathbb{Z}} \in (S^1)^{\mathbb{Z}}$ let $\alpha_\zeta \in \mathrm{Aut}(\mathcal{O}_\infty)$ be the "quasifree" automorphism given by $\alpha_\zeta(s_k) = \zeta_k s_k$ for $k \in \mathbb{Z}$. Let h be the shift.

Here, h is not minimal, but one can restrict to minimal subshifts. Also, other simple separable unital C*-algebras have quasifree automorphisms.

More general families of examples

We choose X, $h\colon X \to X$, a simple separable unital C*-algebra D, and a continuous map $x \mapsto \alpha_x$ from X to $\mathrm{Aut}(D)$. Then $\alpha \in \mathrm{Aut}(C(X, D))$ is $\alpha(a)(x) = \alpha_x(a(h^{-1}(x)))$ for $a \in C(X, D)$ and $x \in X$.

Example

Set $X = (S^1)^n$, let $h\colon X \to X$ be a product of independent irrational rotations, let $D = C^*_{\mathrm{r}}(F_n)$, with unitary generators $u_1, u_2, \ldots, u_n$, and for $\zeta = (\zeta_k)_{1 \leqslant k \leqslant n} \in (S^1)^n$ let $\alpha_\zeta \in \mathrm{Aut}(\mathcal{O}_\infty)$ be the "quasifree" automorphism given by $\alpha_\zeta(u_k) = \zeta_k u_k$ for $k \in \mathbb{Z}$.

Example

Let X be the Cantor set, and let $h\colon X \to X$ be any minimal homeomorphism. Let $n \in \mathbb{Z}_{>0}$. Choose a partition $X = \coprod_{\rho \in S_n} X_\rho$ over ρ in the symmetric group S_n. Let $D = C^*_{\mathrm{r}}(F_n)$ (as before), and $\psi_\rho \in \mathrm{Aut}(D)$ for $\rho \in S_n$ permute the generators: $\psi_\rho(u_k) = u_{\rho(k)}$. For $x \in X_\rho$ set $\alpha_x = \psi_\rho$.

Outline of the ideas

Let X be a compact Hausdorff space, and let $h\colon X \to X$ be a homeomorphism. Recall the *orbit breaking subalgebra*. Let $Y \subset X$ be closed, and let $u \in C^*(\mathbb{Z}, X, h)$ be the standard unitary, coming from the generator of $\mathbb{Z}$. Then, taking $C_0(X \setminus Y)$ to be the functions in $C(X)$ which vanish on Y,

$$C^*(\mathbb{Z}, X, h)_Y = C^*\big(C(X),\ C_0(X \setminus Y)u\big) \subset C^*(\mathbb{Z}, X, h).$$

Assume h is minimal. If $Y \neq \varnothing$ satisfies $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$ (say $Y = \{x_0\}$), then $C^*(\mathbb{Z}, X, h)_Y$ is a centrally large subalgebra of $C^*(\mathbb{Z}, X, h)$ (some definitions below), and some properties of $C^*(\mathbb{Z}, X, h)_Y$ pass to $C^*(\mathbb{Z}, X, h)$, including stable rank one, tracial Z-stability, and strict comparison of positive elements. Moreover, if $\mathrm{int}(Y) \neq \varnothing$, then $C^*(\mathbb{Z}, X, h)_Y$ is a recursive subhomogeneous algebra (defined below), with base spaces closed subsets of X, and if $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$, then $C^*(\mathbb{Z}, X, h)_Y$ is a direct limit of such things. This makes it possible to use direct limit methods to prove properties of $C^*(\mathbb{Z}, X, h)$.

Outline (continued)

For $C(X, D)$, we just put in D everywhere.

Let X be a compact Hausdorff space, let $h \colon X \to X$ be a homeomorphism, and let D be a unital C*-algebra. Let $\alpha \in \operatorname{Aut}(C(X, D))$ lie over h. Again let u be the standard unitary in the crossed product. For $Y \subset X$ closed, define

$$\begin{aligned} C^*\big(\mathbb{Z}, \, C(X, D), \, \alpha\big)_Y &= C^*\big(C(X, D), \, C_0(X \setminus Y, \, D)u\big) \\ &\subset C^*\big(\mathbb{Z}, \, C(X, D), \, \alpha\big). \end{aligned}$$

Assume h is minimal. If $\operatorname{int}(Y) \neq \varnothing$, this is a "recursive subhomogeneous algebra over D" (defined below). If $Y_1 \supset Y_2 \supset \cdots$ are closed subsets of X such that $\bigcap_{n=1}^{\infty} Y_n = Y$, then

$$\begin{aligned} C^*\big(\mathbb{Z}, \, C(X, D), \, \alpha\big)_Y &= \overline{\bigcup_{n=1}^{\infty} C^*\big(\mathbb{Z}, \, C(X, D), \, \alpha\big)_{Y_n}} \\ &\cong \varinjlim C^*\big(\mathbb{Z}, \, C(X, D), \, \alpha\big)_{Y_n}. \end{aligned}$$

Outline (continued)

If $\alpha \in \mathrm{Aut}(C(X, D))$ lies over h, and $Y \subset X$ is closed, then

$$C^*\big(\mathbb{Z},\ C(X, D),\ \alpha\big)_Y = C^*\big(C(X, D),\ C_0(X \setminus Y,\ D)u\big).$$

If $Y_1 \supset Y_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} Y_n = Y$, then

$$C^*\big(\mathbb{Z},\ C(X, D),\ \alpha\big)_Y \cong \varinjlim C^*\big(\mathbb{Z},\ C(X, D),\ \alpha\big)_{Y_n}.$$

If D is simple, $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$, and an additional mild technical condition is satisfied (see below), then $C^*\big(\mathbb{Z},\ C(X, D),\ \alpha\big)_Y$ is centrally large in $C^*\big(\mathbb{Z},\ C(X, D),\ \alpha\big)$.

So in principle one can use direct limit methods to prove properties of $C^*\big(\mathbb{Z},\ C(X, D),\ \alpha\big)$.

Outline (continued)

As a simpler case of recursive subhomogeneous algebras over D and their direct limits, let's briefly look at "homogeneous algebras over D" and their direct limits. A homogeneous algebras over D is of the form

$$C(Z_1, M_{k_1}(D)) \oplus C(Z_2, M_{k_2}(D)) \oplus \cdots \oplus C(Z_n, M_{k_n}(D))$$

for compact metric spaces $Z_1, Z_2, \ldots, Z_n$ and $k_1, k_2, \ldots, k_n \in \mathbb{Z}_{>0}$. (To be consistent with the usual meaning of homogeneity for a C*-algebra, one should allow section algebras of locally trivial bundles here.) To simplify notation in a direct system, let's (unreasonably) assume that each term of the system has only one summand. So we are looking at simple direct limits of direct systems of the form

$$C(X_0, M_{d_0}(D)) \longrightarrow C(X_1, M_{d_1}(D)) \longrightarrow C(X_2, M_{d_2}(D)) \longrightarrow \cdots .$$

In general, very little is known about such direct limits, and we don't yet have anything new here. The methods used for simple AH algebras with slow dimension growth don't apply directly.

Outline (continued)

Under a technical condition, we reduce the study of certain properties of $C^*(\mathbb{Z}, C(X, D), \alpha)$ to the study of the same properties for the recursive subhomogeneous algebra generalization of a direct limit of a system

$$C(X_0, M_{d_0}(D)) \longrightarrow C(X_1, M_{d_1}(D)) \longrightarrow C(X_2, M_{d_2}(D)) \longrightarrow \cdots,$$

assuming the direct limit is simple, and with X_n a closed subset of X for all n.

Known results give some information (also for the recursive subhomogeneous case) in the following situations:

1. D is purely infinite simple (and h is wild). (This doesn't need large subalgebras.)
2. D is Z-stable (and h is wild).
3. X is the Cantor set and D has stable rank one (but is otherwise wild).
4. $X = S^1$, D has stable rank one and real rank zero, and $K_1(D) = 0$ (but D is otherwise wild).

Even with these restrictions, there are still many examples.

Recursive subhomogeneous algebras over D

Definition

Let A, B, and C be C*-algebras, and let $\varphi \colon A \to C$ and $\psi \colon B \to C$ be homomorphisms. Then the *pullback* $A \oplus_{C, \varphi, \psi} B$ (or $A \oplus_C B$ when φ and ψ are understood) is

$$A \oplus_{C, \varphi, \psi} B = \big\{ (a, b) \in A \oplus B \colon \varphi(a) = \psi(b) \big\}.$$

Recursive subhomogeneous algebras over D will then be iterated pullbacks of algebras of the form $C(X, M_d(D))$:

$$C(X, M_d(D))$$

$$C(X_0, M_{d_0}(D)) \oplus_{C(X_1^{(0)}, M_{d_1}(D))} C(X_1, M_{d_1}(D))$$

with $X_1^{(0)} \subset X$ closed, etc., satisfying some conditions.

Taking $D = \mathbb{C}$ in the definition on the next slide gives the original definition of a recursive subhomogeneous algebra.

Recursive subhomogeneous algebras over D: the definition

$$A \oplus_{C,\varphi,\psi} B = \big\{ (a, b) \in A \oplus B \colon \varphi(a) = \psi(b) \big\}.$$

Definition

Let D be a simple unital C*-algebra. The class of *recursive subhomogeneous algebras over D* is the smallest class $\mathcal{R}$ of C*-algebras that is closed under isomorphism and such that:

1. If X is a compact Hausdorff space and $n \geqslant 1$, then $C(X, M_n(D)) \in \mathcal{R}$.
2. If $B \in \mathcal{R}$, X is compact Hausdorff, $n \geqslant 1$, $X^{(0)} \subset X$ is closed (possibly empty), $\varphi \colon B \to C(X^{(0)}, M_n(D))$ is any unital homomorphism (the zero homomorphism if $X^{(0)}$ is empty), and $\rho \colon C(X, M_n(D)) \to C(X^{(0)}, M_n(D))$ is restriction, then

$$B \oplus_{C(X^{(0)}, M_n(D)), \varphi, \rho} C(X, M_n(D)) = \big\{ (b, f) \in B \oplus C(X, M_n(D)) \colon \varphi(b) = f|_{(X^{(0)}} \big\}$$

is in $\mathcal{R}$.

Centrally large subalgebras

Recall Cuntz subequivalence: for $a, b \in M_\infty(A)_+$, we say that a is *Cuntz subequivalent to b in A*, written $a \precsim_A b$, if there is a sequence $(v_n)_{n=1}^{\infty}$ in $M_\infty(A)$ such that $\lim_{n \to \infty} v_n b v_n^* = a$.

Proposition

Let A be a finite infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a unital subalgebra. Assume that for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $r \in A_+ \setminus \{0\}$, and $s \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

1. $g \precsim_B s$ and $g \precsim_A r$.
2. $0 \leqslant g \leqslant 1$.
3. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
4. For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
5. For $j = 1, 2, \ldots, m$ we have $\|g a_j - a_j g\| < \varepsilon$.

Then B is a centrally large subalgebra of A.

Why we need conditions on the automorphism

Let A be an infinite dimensional finite simple unital C*-algebra, and let $B \subset A$ be a unital subalgebra. Then B is centrally large in A if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $r \in A_+ \setminus \{0\}$, and $s \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that

1. $g \precsim_B s$ and $g \precsim_A r$

and some other conditions hold.

We want to take

$$A = C^*\big(\mathbb{Z},\, C(X, D),\, \alpha\big) \qquad \text{and} \qquad B = C^*\big(\mathbb{Z},\, C(X, D),\, \alpha\big)_Y,$$

with $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$.

If $D = \mathbb{C}$, it is known that B is centrally large in A. A commonly used argument with simple crossed products shows that it suffices to take $r, s \in C(X)_+ \setminus \{0\}$ (and $r = s$). For r, the point is that there is $f \in C(X)_+ \setminus \{0\}$ such that $f \precsim_{C*(Z,X,h)} r$. So in the argument we use $f \precsim_{C*(Z,X,h)} r$ in place of r.

Conditions on the automorphism (continued)

We have "error tolerances"

$$r \in C^*(\mathbb{Z},\, C(X,D),\, \alpha)_+ \setminus \{0\} \quad \text{and} \quad s \in \big(C^*(\mathbb{Z},\, C(X,D),\, \alpha)_Y\big)_+ \setminus \{0\}.$$

If $D = \mathbb{C}$, we can reduce to $r = s = f$ with $f \in C(X)_+ \setminus \{0\}$.

In the current situation, the argument for $D = \mathbb{C}$ works if we take $f \in C(X) \subset C(X,D)$, but the "commonly used argument" gives a reduction to $r, s \in C(X,D)_+ \setminus \{0\}$ instead. We can improve it to $r = f_0 \otimes d$ with $f_0 \in C(X)_+ \setminus \{0\}$ and $d \in D_+ \setminus \{0\}$, but we need to show that (just considering the crossed product), given $f_0 \in C(X)_+ \setminus \{0\}$ and $d \in D_+ \setminus \{0\}$, we can find $f_1 \in C(X)_+ \setminus \{0\}$ such that

$$f_1 \otimes 1 \precsim_{C*(\mathbb{Z},\, C(X,D),\, \alpha)} f_0 \otimes d.$$

Conditions on the automorphism (continued)

Suppose $f_0 \in C(X)_+ \setminus \{0\}$ and $d \in D_+ \setminus \{0\}$. We want $f_1 \in C(X)_+ \setminus \{0\}$ such that

$$f_1 \otimes 1 \precsim_{C^*(\mathbb{Z}, C(X,D), \alpha)} f_0 \otimes d.$$

Suppose $\alpha_x = \mathrm{id}_D$ for all $x \in X$. Then $\alpha(a) = a \circ h^{-1}$ for all $a \in C(X, D)$, not just all $a \in C(X)$. Since D is simple, there is $l \in \mathbb{Z}_{>0}$ such that $1 \precsim_D d \oplus d \oplus \cdots \oplus d$ (l copies of d). Now use minimality and freeness to find a nonempty open set $V \subset X$ and $n_1, n_2, \ldots, n_l \in \mathbb{Z}$ such that the sets

$$h^{n_1}\big(\overline{V}\big),\ h^{n_2}\big(\overline{V}\big),\ \ldots,\ h^{n_l}\big(\overline{V}\big)$$

are disjoint and contained in $\{x \in X \colon f_0(x) > 0\}$, and choose $f_1 \in C(X)_+ \setminus \{0\}$ such that $\mathrm{supp}(f_1) \subset V$. Then (with l copies of $f_1 \otimes d$ on the right in the first line)

$$\begin{aligned} f_1 \otimes 1 & \precsim_{C(X,D)} f_1 \otimes d \oplus f_1 \otimes d \oplus \cdots \oplus f_1 \otimes d \\ & \sim_{C^*(\mathbb{Z}, C(X,D), \alpha)} \big[f_1 \circ h^{-n_1} \otimes d\big] \oplus \big[f_1 \circ h^{-n_2} \otimes d\big] \\ & \qquad \oplus \cdots \oplus \big[f_1 \circ h^{-n_l} \otimes d\big] \\ & \precsim_{C(X,D)} f_0 \otimes d. \end{aligned}$$

Conditions on the automorphism (continued)

The calculation from the previous slide:

$$\begin{aligned} f_1 \otimes 1 &\precsim_{C(X,D)} f_1 \otimes d \oplus f_1 \otimes d \oplus \cdots \oplus f_1 \otimes d \\ &\sim_{C^*(\mathbb{Z},\, C(X,D),\, \alpha)} [f_1 \circ h^{-n_1} \otimes d] \oplus [f_1 \circ h^{-n_2} \otimes d] \\ &\qquad\qquad \oplus \cdots \oplus [f_1 \circ h^{-n_l} \otimes d] \\ &\precsim_{C(X,D)} f_0 \otimes d. \end{aligned}$$

It is only because $\alpha_x = \mathrm{id}_D$ for all $x \in X$ that we get

$$f_1 \otimes d \sim_{C^*(\mathbb{Z},\, C(X,D),\, \alpha)} f_1 \circ h^{-n_j} \otimes d.$$

Even for $\alpha = (f \mapsto f \circ h^{-1}) \otimes \gamma$ with $\gamma \in \mathrm{Aut}(D)$, there could be trouble. We need to know that $\{\gamma^n(d) \colon n \in \mathbb{Z}\}$ is "bounded away from zero in $\mathrm{Cu}(D)$". With α coming from $x \mapsto \alpha_x \in \mathrm{Aut}(D)$, the situation is technically more complicated, but in principle very similar.

Conditions on the automorphism (continued)

Recall that $\alpha(a)(x) = \alpha_x(a(h^{-1}(x)))$ for all $a \in C(X, D)$ and $x \in X$, and that D is simple and unital. Let $H \subset \mathrm{Aut}(D)$ be the subgroup generated by $\{\alpha_x : x \in X\}$.

It is enough to to know that for $d \in D_+ \setminus \{0\}$, the set $\{\varphi(d) : \varphi \in H\}$ is "bounded away from zero in $\mathrm{Cu}(D)$". Any of the following is sufficient:

1. α_x is approximately inner in $\mathrm{Aut}(D)$ for all $x \in X$.
2. D has strict comparison of positive elements.
3. D has property (SP) and the order on projections over D is determined by quasitraces.
4. H is contained in a subset of $\mathrm{Aut}(D)$ which is compact in the topology of pointwise norm convergence.
5. D is purely infinite.

(We don't actually use (5).)

We know of no examples in which "boundedness away from zero" fails, but we suppose they must exist, even for H of the form $\{\gamma^n : n \in \mathbb{Z}\}$.

The theorems

Theorem

Let G be a discrete group, and let X be a compact space. Suppose G acts minimally on X and for every finite $S \subset G \setminus \{1\}$, the set $\{x \in X : gx \neq x \text{ for all } g \in S\}$ is dense in X. Let D be a purely infinite simple unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(C(X, D))$ lie over the given action of G on X. Then $C^*_{\mathrm{r}}(G, C(X, D), \alpha)$ is purely infinite simple.

The definition of "lies over" is the obvious generalization of the one already given. No theory of large subalgebras is needed.

In the other theorems, X is an infinite compact metric space, $h \colon X \to X$ is a minimal homeomorphism, D is a simple unital C*-algebra, and $\alpha \in \mathrm{Aut}(C(X, D))$ is an automorphism which lies over h.

Theorem

Assume that D is a simple separable unital Z-stable C*-algebra which has a quasitrace. Then $C^*(\mathbb{Z}, C(X, D), \alpha)$ is tracially Z-stable. If, in addition, D is nuclear, then $C^*(\mathbb{Z}, C(X, D), \alpha)$ is Z-stable.

The theorems (continued)

Recall: X is an infinite compact metric space, $h \colon X \to X$ is a minimal homeomorphism, D is a simple unital C*-algebra, and $\alpha \in \operatorname{Aut}(C(X, D))$ is an automorphism which lies over h.

Theorem

Assume that X is the Cantor set. Further assume that one of the following conditions holds:

1. α_x is approximately inner in $\operatorname{Aut}(D)$ for all $x \in X$.
2. D has strict comparison of positive elements.
3. D has property (SP) and the order on projections over D is determined by quasitraces.
4. H is contained in a subset of $\operatorname{Aut}(D)$ which is compact in the topology of pointwise norm convergence.

If $\operatorname{tsr}(D) = 1$, then $\operatorname{tsr}\big(C^*(\mathbb{Z}, C(X, D), \alpha)\big) = 1$, and if also $\operatorname{RR}(D) = 0$, then $\operatorname{RR}\big(C^*(\mathbb{Z}, C(X, D), \alpha)\big) = 0$.

The theorems (continued)

Recall: X is an infinite compact metric space, $h\colon X \to X$ is a minimal homeomorphism, D is a simple unital C*-algebra, and $\alpha \in \mathrm{Aut}(C(X, D))$ is an automorphism which lies over h.

Theorem

Assume that $\dim(X) \leqslant 1$, that D has stable rank one and real rank zero, and that $K_1(D) = 0$. Further assume that one of the following conditions holds:

1. α_x is approximately inner in $\mathrm{Aut}(D)$ for all $x \in X$.
2. The order on projections over D is determined by quasitraces.
3. H is contained in a subset of $\mathrm{Aut}(D)$ which is compact in the topology of pointwise norm convergence.

Then $C^* \big(\mathbb{Z}, \, C(X, D), \, \alpha \big)$ has stable rank one.

Example

Set $X_0 = (S^1)^{\mathbb{Z}}$. Let $h_0 \colon X_0 \to X_0$ be the (forwards) shift. Let $D = \bigotimes_{n \in \mathbb{Z}_{>0}} M_2$ be the 2^∞ UHF algebra. Choose a bijection $\sigma \colon \mathbb{Z}_{>0} \to \mathbb{Z}$, and for $\zeta = (\zeta_n)_{n \in \mathbb{Z}} \in X_0$ define

$$\alpha_\zeta = \bigotimes_{n \in \mathbb{Z}_{>0}} \mathrm{Ad} \left(\begin{pmatrix} 1 & 0 \\ 0 & \zeta_{\sigma(n)} \end{pmatrix} \right) \in \mathrm{Aut}(D).$$

Then $\zeta \mapsto \alpha_\zeta^{(0)}$ is continuous.

The homeomorphism h_0 has mean dimension 1 but isn't minimal. So use known methods to choose a closed invariant subset $X \subset X_0$ such that $h = h_0|_X$ has strictly positive mean dimension. Then h and $\zeta \mapsto \alpha_\zeta$ define an automorphism α of $C(X, D)$.

The crossed product $C^*\big(\mathbb{Z}, C(X, D), \alpha\big)$ is Z-stable, even though $C^*(\mathbb{Z}, X, h)$ surely is not.

(Joint work with Hirshberg proves that $C^*(\mathbb{Z}, X, h)$ is not Z-stable for a modification using $(S^1)^2$.)

Notation for full and reduced C*-algebras of free groups

Several examples use $C^*_{\mathrm{r}}(F_n)$ or $C^*(F_n)$ with $n \in \mathbb{Z}_{>0} \cup \{\infty\}$. We let $u_1, u_2, \ldots, u_n \in C^*_{\mathrm{r}}(F_n)$ or $C^*(F_n)$ be the "standard" unitaries, obtained as the images of the standard generators of F_n. Use analogous notation for $C^*_{\mathrm{r}}(F_\infty)$ and $C^*(F_\infty)$, but indexing the generators as $\ldots, u_{-1}, u_0, u_1, u_2, \ldots$.

There are "quasifree" automorphisms, given, for example, on $C^*_{\mathrm{r}}(F_n)$ by $\varphi_{\zeta_1,\zeta_2,\ldots,\zeta_n}(u_k) = \zeta_k u_k$ for $\zeta_1, \zeta_2, \ldots, \zeta_n \in S^1$ and $k = 1, 2, \ldots, n$.

There is also a reduced free shift $\sigma \in \mathrm{Aut}(C^*_{\mathrm{r}}(F_\infty))$, determined by $\sigma(u_n) = u_{n+1}$ for $n \in \mathbb{Z}$.

Example

Let X (shift invariant subspace of $(S^1)^{\mathbb{Z}}$), h (with strictly positive mean dimension), D (the 2^∞ UHF algebra), and $\zeta \mapsto \alpha_\zeta \in \mathrm{Aut}(D)$ be as in the previous example. Define $E = D \otimes C^*_{\mathrm{r}}(F_\infty)$. For $\zeta \in X$ define $\beta_\zeta = \alpha_\zeta \otimes \varphi_\zeta$. (Recall that φ_ζ is the quasifree automorphism $u_k \mapsto \zeta_k u_k$.)

Then $C^*\big(\mathbb{Z}, C(X, E), \beta\big)$ is tracially Z-stable since E is.

It also has stable rank one. To see this: E is exact and D-stable, so has strict comparison of positive elements. Choose any one point subset $Y \subset X$. Then $C^*\big(\mathbb{Z}, C(X, E), \beta\big)_Y$ is simple and Z-stable. So it has stable rank one by a result of Rørdam. The algebra $C^*\big(\mathbb{Z}, C(X, E), \beta\big)_Y$ is centrally large in $C^*\big(\mathbb{Z}, C(X, E), \beta\big)$, so $C^*\big(\mathbb{Z}, C(X, E), \beta\big)$ has stable rank one.

We don't know whether $C^*\big(\mathbb{Z}, C(X, E), \beta\big)$ is Z-stable. We also don't know whether tracial Z-stability implies stable rank one, although this may well be true.

Example

Let X be the Cantor set and let h be an arbitrary minimal homeomorphism of X. Let $\sigma \in \mathrm{Aut}(C^*_{\mathrm{r}}(F_\infty))$ be the reduced free shift. Let $\alpha \in \mathrm{Aut}\big(C(X) \otimes C^*_{\mathrm{r}}(F_\infty)\big)$ be the tensor product of the automorphism $f \mapsto f \circ h^{-1}$ of $C(X)$ and σ.

The algebra $C^*_{\mathrm{r}}(F_\infty)$ is known to have stable rank one (Dykema-Haagerup-Rørdam) and strict comparison of positive elements (Robert), so we can use the theorem for the Cantor set to see that $C^*\big(\mathbb{Z}, \, C(X) \otimes C^*_{\mathrm{r}}(F_\infty), \, \alpha\big)$ has stable rank 1.

Apparently no previously known results give anything about this crossed product. In particular, knowing that the action has the Rokhlin property doesn't seem to help with proving stable rank one.

For n finite, it isn't known whether $C^*_{\mathrm{r}}(F_n)$ has strict comparison of positive elements.

Recall that a Denjoy homeomorphism of S^1 is a nonminimal homeomorphism of S^1 with an irrational rotation number θ. It comes with a surjective equivariant map ζ to S^1 with rotation by $2\pi\theta$. It has a unique minimal set X, homeomorphic to the Cantor set. We call its restriction to X a *restricted Denjoy homeomorphism*. We also restrict ζ to X.

Example

Fix $n \in \{2, 3, \ldots, \infty\}$. Fix $\theta_1, \theta_2, \ldots, \theta_n \in \mathbb{R}$ (or $\theta_1, \theta_2, \ldots \in \mathbb{R}$) such that $1, \theta_1, \theta_2, \ldots, \theta_n$ (or $1, \theta_1, \theta_2, \ldots$) are linearly independent over $\mathbb{Q}$. For $k = 1, 2, \ldots, n$ (or $k = 1, 2, \ldots$) let $h_k \colon X_k \to X_k$ be a restricted Denjoy homeomorphism with rotation number $\theta_k \in \mathbb{R} \setminus \mathbb{Q}$. Take $X = \prod_{k=1}^{n} X_k$, and let $h \colon X \to X$ act as h_k on the k-th factor. Then h is minimal. (There seems to be no proof in the literature, so we proved it.)

Let $\zeta_k \colon X_k \to S^1$ be the associated map as above. Using quasifree automorphisms of $C^*_{\mathrm{r}}(F_n)$, take $\alpha_x(u_k) = \zeta_k(x_k) u_k$. This gives $\alpha \in \operatorname{Aut}\big(C(X, \, C^*_{\mathrm{r}}(F_n))\big)$. Then $C^*\big(\mathbb{Z}, \, C(X, \, C^*_{\mathrm{r}}(F_n)), \, \alpha\big)$ has stable rank one, using our Cantor set theorem, since $\{\alpha_x \colon x \in X\}$ is contained in a compact subgroup of $\operatorname{Aut}\big(C^*_{\mathrm{r}}(F_n)\big)$ and $\operatorname{tsr}\big(C^*_{\mathrm{r}}(F_n)\big) = 1$ (Dykema-Haagerup-Rørdam).

An example with $X = S^1$

We sketch the construction of an example with $X = S^1$. We adapt a construction of Dadarlat.

Let $(\pi_1, \pi_2, \ldots)$ be a sequence of unital finite dimensional representations of $C^*(F_2)$ such that, for every n, the representation $\bigoplus_{k=n}^{\infty} \pi_k$ is faithful. Let $u_1, u_2 \in C^*(F_2)$ be the standard unitary generators of $C^*(F_2)$ (as before). Let $\varphi_1, \varphi_2, \varphi_3 \in \mathrm{Aut}(C^*(F_2))$ be the automorphisms determined by

$$\varphi_1(u_1) = u_1^{-1} \quad \text{and} \quad \varphi_1(u_2) = u_2,$$

$$\varphi_2(u_1) = u_1 \quad \text{and} \quad \varphi_2(u_2) = u_2^{-1},$$

and

$$\varphi_3(u_1) = u_1^{-1} \quad \text{and} \quad \varphi_3(u_2) = u_2^{-1}.$$

For $n \in \mathbb{Z}_{>0}$ define

$$D_n = M_{l(n)} \otimes M_{l(n-1)} \otimes \cdots \otimes M_{l(1)} \otimes C^*(F_2)$$

(with matrix sizes $l(m)$ chosen to make the formulas below work).

An example with $X = S^1$ (continued)

$\pi_1, \pi_2, \ldots$ are unital finite dimensional representations of $C^*(F_2)$.
$\varphi_1, \varphi_2, \varphi_3 \in \mathrm{Aut}(C^*(F_2))$ are

$$\varphi_1(u_1) = u_1^{-1};\ \varphi_1(u_2) = u_2, \quad \varphi_2(u_1) = u_1,\ \varphi_2(u_2) = u_2^{-1};$$
$$\varphi_3(u_1) = u_1^{-1},\ \varphi_3(u_2) = u_2^{-1}.$$

$l(m)$ are appropriate matrix sizes.
Define

$$\gamma_{n,n-1} \colon D_{n-1} \to D_n = M_{l(n)}(D_{n-1})$$

by, for $a \in D_{n-1} = M_{r(n-1)} \otimes C^*(F_2)$,

$$\gamma_{n,n-1}(a) = \mathrm{diag}\big(a,\ (\mathrm{id}_{M_{r(n-1)}} \otimes \varphi_1)(a),\ (\mathrm{id}_{M_{r(n-1)}} \otimes \varphi_2)(a),$$
$$(\mathrm{id}_{M_{r(n-1)}} \otimes \varphi_3)(a),\ (\pi_n \otimes \mathrm{id}_{M_{r(n-1)}}(a) \otimes 1\big).$$

Methods of Dadarlat show that the direct limit D is simple and has tracial rank zero. So D has stable rank one, real rank zero, and strict comparison of positive elements. The maps φ_j ensure that $K_1(D) = 0$. The algebra D isn't even exact, since it contains a copy of $C^*(F_2)$. Methods of Niu and Wang will probably show that D is not Z-stable.

An example with $X = S^1$ (continued)

The map

$$\gamma_{n,n-1} \colon D_{n-1} \to D_n = M_{l(n)}(D_{n-1})$$

is

$$\gamma_{n,n-1}(a) = \operatorname{diag}\big(a, \, (\mathrm{id}_{M_{r(n-1)}} \otimes \varphi_1)(a), \, (\mathrm{id}_{M_{r(n-1)}} \otimes \varphi_2)(a), \\ (\mathrm{id}_{M_{r(n-1)}} \otimes \varphi_3)(a), \, (\pi_n \otimes \mathrm{id}_{M_{r(n-1)}}(a) \otimes 1\big),$$

and D is the direct limit.

For $\zeta \in S^1$ define inductively unitaries $u_n(\zeta) \in D_n$ as follows. Set $u_0(\zeta) = 1$, and, given $u_{n-1}(\zeta) \in D_{n-1}$, set

$$u_n(\zeta) = \big(\operatorname{diag}(\zeta, 1, 1, \ldots, 1) \otimes 1_{D_{n-1}}\big) \gamma_{n,n-1}(u_{n-1}(\zeta)) \in M_{l(n)} \otimes D_{n-1} = D_n.$$

One can check that conjugation by these unitaries gives an action $\zeta \to \alpha_\zeta$ of S^1 on D. Let $h \colon S^1 \to S^1$ be an irrational rotation. Use it and $\zeta \mapsto \alpha_\zeta$ to get $\alpha \in \operatorname{Aut}(C(S^1, D))$.

The crossed product $C^*\big(\mathbb{Z}, \, C(S^1, D), \, \alpha\big)$ has stable rank one by the theorem for 1-dimensional spaces.

Example

Let B be the 2^∞ UHF algebra, and let τ be its (unique) tracial state. Let tr be the tracial state on M_2, and let ρ be the state on M_2 given by $\rho(x) = \mathrm{tr}\big(\mathrm{diag}\big(\frac{1}{3}, \frac{2}{3}\big)x\big)$. Let D be the reduced free product $D = (B \otimes M_2) \star_{\mathrm{r}} C([0,1])$, taken with respect to the state $\tau \otimes \rho$ on $B \otimes M_2$ and the Lebesgue measure state on $C([0,1])$. With work, one can show that A is purely infinite and simple but not $\mathcal{Z}$-stable.

Let X (shift invariant subspace of $(S^1)^{\mathbb{Z}}$) and h (with strictly positive mean dimension) be as in the first example. Let $\sigma \colon \mathbb{Z}_{>0} \to \mathbb{Z}$ be a bijection. For $\zeta = (\zeta_n)_{n \in \mathbb{Z}} \in X$ take α_ζ to be the free product automorphism which is the identity on $C([0,1])$ and which is the infinite tensor product

$$\left[\bigotimes_{n \in \mathbb{Z}_{>0}} \mathrm{Ad}\left(\begin{pmatrix} 1 & 0 \\ 0 & \zeta_{\sigma(n)} \end{pmatrix} \right) \right] \otimes \mathrm{Ad}\left(\begin{pmatrix} \zeta_{\sigma(1)} & 0 \\ 0 & \zeta_{\sigma(2)} \end{pmatrix} \right) \in \mathrm{Aut}(B \otimes M_2)$$

on $B \otimes M_2$. Then $C^*(\mathbb{Z}, C(X, D), \alpha)$ is purely infinite and simple.

$C^*(\mathbb{Z}, C(X, D), \alpha)$ may well not be $\mathcal{Z}$-stable.