Aperiodicity and related properties for crossed product inclusions

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Motivation: Ideal structure of crossed products

Basic situation

G discrete group

A C*-algebra

α group action, i.e., homomorphism $G \to \mathrm{Aut}(A)$

B reduced crossed product $A \rtimes_{\mathrm{r}} G$

Questions

When is B simple?
When it is not simple,
what can we say about the ideal lattice of B?

Kishimoto's Theorem

Theorem (Kishimoto 1981)

Let A be simple. If the automorphisms α_g for $g \in G \setminus \{1\}$ are outer, then $A \rtimes_{\mathrm{r}} G$ is simple.

Lemma (Key step in Kishimoto's proof)

In the situation above, for any $g \in G \setminus \{1\}$,
$x \in A$, $\varepsilon > 0$, and non-zero hereditary subalgebra $D \subseteq A$,
there is $a \in D$ with $0 \le a$, $\|a\| = 1$ and $\|ax\alpha(a)\| < \varepsilon$.

Definition (Kwaśniewski–M)

A normed A-bimodule X is aperiodic if,
for each $x \in X$, $\varepsilon > 0$, and non-zero hereditary subalgebra $D \subseteq A$,
there is $a \in D_1^+$ with $\|axa\| < \varepsilon$.

Definition

A C^*-inclusion $A \subseteq B$ is aperiodic if
B/A with the quotient norm is an aperiodic A-bimodule.

Aperiodicity and Kishimoto's condition

Key Lemma about Aperiodicity

Let X_i for $i \in I$ and Y be normed A-bimodules.
Let $f_i \colon X_i \to Y$ be bounded A-bimodule maps.
Assume that $\sum f_i(X_i)$ is dense in Y.
If X_i is aperiodic for all $i \in I$, then so is Y.

Theorem

$A \subseteq A \rtimes_{\mathrm{r}} G$ is aperiodic if and only if, for all $g \in G \setminus \{1\}$,
A with the bimodule structure $a \bullet b \bullet c = ab\alpha_g(c)$ is aperiodic,
if and only if, for all $g \in G \setminus \{1\}$,
α_g satisfies Kishimoto's condition.

Ideal detection

Definition

$A \subseteq B$ detects ideals if $J \cap A \neq 0$ for any ideal $J \subseteq B$ with $J \neq 0$.

Definition

Call A B-simple if $BIB = B$ for any non-zero ideal $I \subseteq A$.

Lemma

B simple $\iff$ $A \subseteq B$ detects ideals and A is B-simple.

Proof of $\Longleftarrow$.

Let $J \subseteq B$ be a non-zero ideal.
Then $J \cap A$ is non-zero because A detects ideals in B.
Then $B(J \cap A)B = B$ because A is B-simple.
Always $J = BJB \supseteq B(J \cap A)B$. So $J = B$. □

Lemma

For a crossed product B and an ideal I in A,
BIB is the smallest invariant ideal containing I.
B-Simple means minimal.

Aperiodicity and detection of ideals

Theorem

Let $A \subseteq B$ be aperiodic. Let $J \subseteq B$ be an ideal.
$J \cap A = 0 \iff J$ *is aperiodic.*

Proof.

Let J be aperiodic. Then so is $J \cap A \subseteq J$.
The latter is also an ideal in A. The only aperiodic ideal in A is 0.
Let $J \cap A = 0$. Then the map $J \to B/A$ is isometric!
Since B/A is aperiodic, so is $J \subseteq B/A$. □

Corollary

Let G act aperiodically on A.
An ideal in $A \rtimes G$ is aperiodic if it is contained in the kernel of the projection $\Lambda\colon A \rtimes G \to A \rtimes_{\mathrm{r}} G$.

Proof.

$\ker \Lambda \cap A = 0$. □

Aperiodicity and detection of ideals II

Theorem

Let G act aperiodically on A.
An ideal in $A \rtimes G$ is aperiodic (if and) only if it is contained in the kernel of the projection $\Lambda\colon A \rtimes G \to A \rtimes_{\mathrm{r}} G$.

Proof.

- Let J be aperiodic.
- The canonical expectation $E\colon A \rtimes G \to A$ maps J to an aperiodic ideal in A.
- The only aperiodic ideal in A is 0.
- So $E|_J = 0$.
- Then $J \subseteq \ker \Lambda$. □

Corollary

Let G act aperiodically on A. Then A detects ideals in $A \rtimes_{\mathrm{r}} G$.

More general crossed products

- So far, we only discussed group actions by automorphisms.
- Our main theorem works, however, for all C^*-inclusions with a conditional expectation:

Theorem

Let $A \subseteq B$ be a C^*-inclusion.
Let $E\colon B \to A$ be a conditional expectation.
Let $N \subseteq B$ be the largest two-sided ideal with $E|_N = 0$. Assume that B/A is an aperiodic A-bimodule.
Then an ideal J in B satisfies $J \cap A = 0$ if and only if $J \subseteq N$.
Therefore, A detects ideals in B/N.

More general group actions

- A twisted action only satisfies $\alpha_g \alpha_h = \mathrm{Ad}(U_{g,h})\alpha_{g\cdot h}$ with unitary multipliers $U_{g,h}$ subject to a cocycle condition.
- In a partial action, each α_g is only an isomorphism between ideals in A, and $\alpha_g \alpha_h$ is contained in $\alpha_{g\cdot h}$.
- In a Fell bundle, each α_g is replaced by a Hilbert bimodule A_g (partial Morita equivalence), and we require Hilbert bimodule maps $A_g \otimes_A A_h \to A_{gh}$, subject to associativity.
- For all these kinds of actions, there is a crossed product B.
  It carries a conditional expectation $E\colon B \to A$,
  and B/N is the reduced crossed product.
- There is a canonical G-grading $B = \bigoplus_{g\in G} B_g$.
- The inclusion $A \subseteq B$ is aperiodic if and only if the normed bimodules B_g are aperiodic for all $g \in G \setminus \{1\}$.

C^*-Algebras for Hausdorff, étale, locally compact groupoids

- Let G be a Hausdorff, locally compact, étale groupoid.
- Let X be its object space.
- Restriction to X defines a conditional expectation $E\colon C^*(G) \to C_0(X) \subseteq C^*(G)$.
- The quotient B/N defined by E is the reduced crossed product.
- $C^*(G)$ is spanned by subspaces $C_0(U)$ for bisections $U \subseteq G$.
- The inclusion is aperiodic if and only if G is topologically free: if U is a bisection with $\emptyset \neq U \subseteq G \setminus X$, then $r|_U \neq s|_U$.

Theorem

If G is topologically free, then $C_0(X)$ detects ideals in $C^*_r(G)$.

How about non-Hausdorff groupoids?

Theorem (Skandalis 1991)

There is a minimal, topologically free, non-Hausdorff, locally compact, étale groupoid for which $\mathrm{C}^*_\mathrm{r}(G)$ *is not simple.*

Theorem (Exel–Pitts, Kwaśniewski–M 2019)

For each topologically free, non-Hausdorff, locally compact, étale groupoid G*, there is an essential groupoid* C^**-algebra* $\mathrm{C}^*_\mathrm{ess}(G)$ *such that* $\mathrm{C}_0(X) \subseteq \mathrm{C}^*_\mathrm{ess}(G)$ *detects ideals.*

Anti-Aperiodic inclusions

Proposition

Let $A \subseteq B$ be a C^*-inclusion.
There is a largest A-subbimodule $M_{\mathrm{ape}} \subseteq B$ that is aperiodic.
It is norm closed and closed under taking adjoints.

Definition

Call $A \subseteq \tilde{A}$ anti-aperiodic if any aperiodic A-bimodule $M \subseteq \tilde{A}$ is 0.

Proposition (Kwaśniewski–M 2019)

The inclusion $A \subseteq A$ is anti-aperiodic.
So is the inclusion of A into its local multiplier algebra.

Theorem (Kwaśniewski–M 2020)

The inclusion of A into its injective hull is anti-aperiodic.

Aperiodicity and unique pseudo-expectations

Definition

A generalised expectation for $A \subseteq B$ is a completely positive contraction $E\colon B \to \tilde{A}$ for some $A \subseteq \tilde{A}$ with $E|_A = \mathrm{id}_A$.

Any generalised expectation is an A-bimodule map.

Proposition

Let $A \subseteq B$ be aperiodic and $A \subseteq \tilde{A}$ be anti-aperiodic. There is at most one generalised expectation $E\colon B \to \tilde{A}$.

Proof.

Let $E, E'\colon B \to \tilde{A}$ be two generalised expectations.
Then $E - E'\colon B/A \to \tilde{A}$ is an A-bimodule map. Its image is an aperiodic A-bimodule in $\tilde{A}$. This is 0 by assumption. □

Corollary (unique pseudo-expectation)

Let $A \subseteq B$ be aperiodic. There is exactly one generalised expectation from B to the injective hull of A.

Aperiodicity and detection of ideals

Theorem

Let $A \subseteq B$ be aperiodic. Let $E\colon B \to I(A)$ be the unique expectation to the injective hull. Let $J \subseteq B$ be an ideal.
Then $J \cap A = 0$ if and only if J is aperiodic, if and only if $E|_J = 0$.
Let $N \subseteq B$ be the largest two-sided ideal on which E vanishes.
Then A detects ideals in B/N.
And N is the unique ideal in B with this property.

Reduced and essential groupoid C*-algebras

- Let $A = \mathrm{C}_0(X)$ be commutative.
- The injective hull $I(\mathrm{C}_0(X))$ is isomorphic to the quotient of the C*-algebra $\mathrm{Bor}(X)$ of Borel functions on X by the ideal of functions that vanish outside a meagre subset.
- Restriction to the units gives a generalised expectation $E_0\colon \mathrm{C}^*(G) \to \mathrm{Bor}(X)$.
- The reduced groupoid C*-algebra is the quotient $\mathrm{C}^*(G)/N_0$, where N_0 is the largest two-sided ideal killed by E_0.
- The theorem above applies to the composite E of E_0 with the quotient map to $I(\mathrm{C}_0(X))$.
- The essential groupoid C*-algebra is the quotient of $\mathrm{C}^*(G)$ by the largest two-sided ideal killed by E.

Theorem (Kwaśniewski–M 2019)

$\mathrm{C}_0(X)$ detects ideals in $\mathrm{C}^*_{\mathrm{ess}}(G)$ $\iff$ G is topologically free.

The dual groupoid

$\widehat{A}$ {irreducible representations} / unitary equivalence.

$\widehat{\alpha}$ induced action of G on $\widehat{A}$ by homeomorphisms.

Definition

The transformation groupoid $\widehat{A} \rtimes G$ is called dual groupoid.
This is also defined for Fell bundles over inverse semigroups or over étale groupoids.

Definition (Properties of an étale groupoid $\mathcal{G}$)

effective	$U \subseteq \mathcal{G}$ is open, $r\|_U = s\|_U \Longrightarrow U \subseteq \mathcal{G}^0$
topologically free	$U \subseteq \mathcal{G} \setminus \mathcal{G}^0$ is open, $r\|_U = s\|_U \Longrightarrow U = \emptyset$
topologically principal	the set of $x \in \mathcal{G}^0$ for which there is $g \in \mathcal{G} \setminus \mathcal{G}^0$ with $r(g) = x = s(g)$ has empty interior

Lemma

effective $\Longrightarrow$ *topologically free* $\Longleftarrow$ *topologically principal*
with $\Longleftrightarrow$ *twice if* $\mathcal{G}$ *is Hausdorff, second countable.*

Extension of pure states

extension property	any pure state on A extends uniquely to a state on B (necessarily pure)
almost extension property	the set of pure states on A that extend uniquely to a state on B is weak-$*$-dense in the set of pure states on A

Theorem (Kwaśniewski–M, following Akemann–Weaver)

Let ω be a pure state on A. TFAE:

- *ω extends uniquely to $A \rtimes_{\mathrm{r}} G$*
- *ω extends uniquely to $A \rtimes G$*
- *the class of GNS representation of ω has trivial isotropy in $\widehat{A} \rtimes G$.*

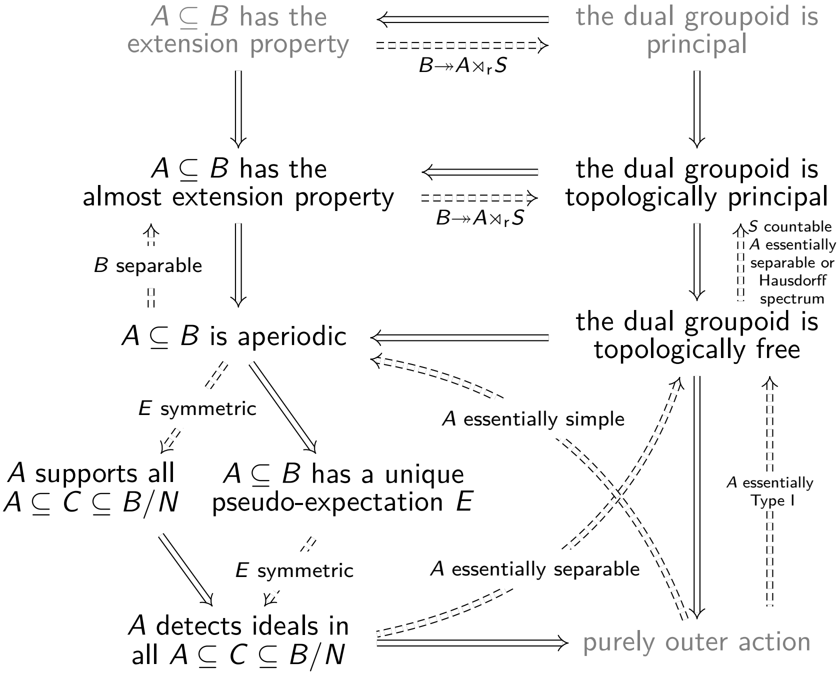
Extension of pure states II

Corollary

TFAE:

- $A \subseteq A \rtimes_{\mathrm{r}} G$ *has the almost extension property*
- $A \subseteq A \rtimes G$ *has the almost extension property*
- *the dual groupoid* $\widehat{A} \rtimes G$ *is topologically principal.*

This remains true for Fell bundles over inverse semigroups or étale groupoids, even for the essential section C^*-algebra.



A ⊆ B has the extension property
the dual groupoid is principal
B↠A⋊rS
A ⊆ B has the almost extension property
the dual groupoid is topologically principal
B↠A⋊rS
B separable
S countable A essentially separable or Hausdorff spectrum
A ⊆ B is aperiodic
the dual groupoid is topologically free
E symmetric
A essentially simple
A supports all A ⊆ C ⊆ B/N
A ⊆ B has a unique pseudo-expectation E
A essentially Type I
E symmetric
A essentially separable
A detects ideals in all A ⊆ C ⊆ B/N
purely outer action