Dynamical criteria towards classifiable transformation group C*-algebras

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Objects of interest: Crossed products $\mathcal{C}(X) \rtimes_\alpha \Gamma$ associated to a topological dynamical system, where

- X is a compact metric space
- Γ is a countable discrete group (usually Γ is amenable)
- $\Gamma \curvearrowright \mathcal{C}(X)$ is induced by an action $\alpha : \Gamma \curvearrowright X$ via homeomorphisms.

Definition (crossed products)

Let $\alpha : \Gamma \curvearrowright A$ be an action on a unital C^*-algebra. Suppose we have some faithful representation $\pi : A \to \mathcal{B}(\mathcal{H})$. Then we consider the unitary representation $u : \Gamma \to \mathcal{U}(\ell^2(\Gamma, \mathcal{H}))$ via $u_g(\xi)(h) = \xi(g^{-1}h)$ and the representation $\pi^\alpha : A \to \mathcal{B}(\ell^2(\Gamma, \mathcal{H}))$ given by

$$\pi^\alpha(a)(\xi)(h) = \alpha_{h^{-1}}(a)\big[\xi(h)\big].$$

The C^*-algebra generated by $u(\Gamma)$ and $\pi^\alpha(A)$ is defined as the reduced crossed product $A \rtimes_{r,\alpha} \Gamma$.

Inside the crossed product one can observe the relation $u_g a u_g^* = \alpha_g(a)$.

Remark

There is also a universal crossed product $A \rtimes_\alpha \Gamma$, which is the universal C^*-algebra generated by a copy of A, the image of a unitary representation $u : \Gamma \to \mathcal{U}(A \rtimes_\alpha \Gamma)$, and the relation $u_g a u_g^* = \alpha_g(a)$ for all $g \in \Gamma$ and $a \in A$.

We will henceforth restrict ourselves to the case where Γ is amenable. In that case it turns out that $A \rtimes_\alpha \Gamma = A \rtimes_{r,\alpha} \Gamma$, so there is no ambiguity involved. Moreover:

Theorem

If Γ is amenable, then A is nuclear if and only if $A \rtimes_\alpha \Gamma$ is nuclear for every (or some) action $\alpha : \Gamma \curvearrowright A$.

The specific crossed products of the form $\mathcal{C}(X) \rtimes \Gamma$ are called transformation group C^*-algebras.

Question

Is there a one-to-one correspondence between certain properties of the topological action $\Gamma \curvearrowright X$ and certain properties of the crossed product $\mathcal{C}(X) \rtimes \Gamma$?

Question

Are there natural criteria on an action $\Gamma \curvearrowright X$ to ensure that $\mathcal{C}(X) \rtimes \Gamma$ is classifiable in the sense of Elliott? Is there a way to topologically characterize when this happens?

Remark (not really a definition)

There is a C^*-algebra $\mathcal{Z}$, called the Jiang–Su algebra, which is separable, infinite-dimensional, unital, monotracial and satisfies $\mathcal{Z} \sim_{KK} \mathbb{C}$.

Definition

A C^*-algebra A is $\mathcal{Z}$-stable if $A \cong A \otimes \mathcal{Z}$.

This property has turned out to be vital for classification, as a reasonable invariant will usually not be able to distinguish between A and $A \otimes \mathcal{Z}$.

On the other hand, $\mathcal{Z}$-stability is indeed quite common and automatic in many cases of interest.

Theorem (Winter, CETWW, many others)

A separable simple nuclear non-elementary C^*-algebra A is $\mathcal{Z}$-stable if and only if A has finite nuclear dimension.

Although we won't define what finite nuclear dimension means, it is a strengthening of nuclearity generalizing finite covering dimension.

The recent highlight of the Elliott program is given by the following result.

Theorem (Elliott–Gong–Lin–Niu, Tikuisis–White–Winter, ...)

Let A and B be separable, unital, simple, nuclear, $\mathcal{Z}$-stable C^*-algebras that satisfy the UCT.[1] If A and B agree on K-theory and traces, then A and B are isomorphic.

Corollary (Elliott)

C^*-algebras with the above properties can be expressed as inductive limits of subhomogeneous algebras of topological dimension ≤ 2.

[1] This is a technical assumption which may or may not be automatic with nuclearity.

Let's start with some more basic properties of crossed products:

Theorem

Suppose Γ *is amenable and* $\alpha : \Gamma \curvearrowright X$ *is an action. Then* $\mathcal{C}(X) \rtimes_\alpha \Gamma$ *is simple if and only if* α *is*

- *minimal: all orbit maps* $[g \mapsto \alpha_g(x)]$ *have dense image.*
- *topologically free: the points* $x \in X$ *for which the orbit map* $[g \mapsto \alpha_g(x)]$ *is injective are topologically generic.*

Unfortunately, the current techniques do not allow us to say much about the fine structure of crossed products for general topologically free actions. We therefore will focus our attention on actions that are *free*, i.e., where all orbit maps are injective.

Theorem (Tu)

If Γ is amenable, then all transformation group C^*-algebras $\mathcal{C}(X) \rtimes \Gamma$ satisfy the UCT.

Corollary

For free minimal actions $\alpha : \Gamma \curvearrowright X$, the crossed products $\mathcal{C}(X) \rtimes_\alpha \Gamma$ are separable, unital, simple, nuclear C^*-algebras satisfying the UCT.

Question

For free minimal actions $\Gamma \curvearrowright X$, when is the crossed product $\mathcal{C}(X) \rtimes \Gamma$ Jiang–Su stable?

After initial work of Putnam, Elliott–Evans and later by Lin–Phillips that attacked the classification question for crossed products head-on, the study of crossed products from the point of view of regularity properties began more recently.

Theorem (Toms–Winter)

For a minimal homeomorphism $T : X \to X$ on a finite-dimensional compact metric space with $|X| = \infty$, the crossed product $\mathcal{C}(X) \rtimes_T \mathbb{Z}$ is $\mathcal{Z}$-stable and has finite nuclear dimension.

The method for proving this result relies on the careful construction of a suitable subalgebra $B \subset \mathcal{C}(X) \rtimes_T \mathbb{Z}$ so that the structural properties of the crossed product can be reduced to those of B. On the other hand, the construction works in such a way that B can be shown to have these properties fairly directly. This idea goes back to work of Putnam and Lin–Phillips[2] and has since been fleshed out in Phillips' theory of *large subalgebras*.

[2] Not to be confused with the "Lin–Phillips" mentioned before. The first author is Q. Lin, not H. Lin.

Theorem (Hirshberg–Winter–Zacharias, a bit later)

For a minimal homeomorphism $T : X \to X$ on a finite-dimensional compact metric space with $|X| = \infty$, the crossed product $\mathcal{C}(X) \rtimes_T \mathbb{Z}$ has finite nuclear dimension.

Although this result is obviously the same as on the previous slide, the method was new and different, based on the concept of Rokhlin dimension for $\mathbb{Z}$-actions on C^*-algebras.

It is not worth it to discuss Rokhlin dimension in detail, but the main point of the theory is that one tends to have the implication

$$\dim_{\mathrm{nuc}}(A) < \infty \text{ and } \dim_{\mathrm{Rok}}(\alpha : \Gamma \curvearrowright A) < \infty \implies \dim_{\mathrm{nuc}}(A \rtimes_\alpha \Gamma) < \infty.$$

It turned out that this kind of method was comparably more ripe for subsequent generalizations, in particular because Rokhlin dimension had the potential to be observed directly at the level of the homeomorphism $T : X \to X$.

Theorem (S)

For any free minimal action $\alpha : \mathbb{Z}^m \curvearrowright X$ on a finite-dimensional compact metric space, the crossed product $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}^m$ has finite nuclear dimension.

This approach included a rather straightforward generalization of Rokhlim dimension for $\mathbb{Z}^m$-actions, and paired it with new insights about the topological dynamical systems. It boils down to the induced action $\mathbb{Z}^m \curvearrowright \mathcal{C}(X)$ having finite Rokhlin dimension, which is achieved by generalizing techniques of Gutman and Lindenstrauss in the context of the so-called *marker property* for homeomorphisms.

Theorem (S–Wu–Zacharias, Bartels)

Let Γ be a finitely generated, virtually nilpotent group. For any free minimal action $\alpha : \Gamma \curvearrowright X$ on a finite-dimensional compact metric space, the crossed product $\mathcal{C}(X) \rtimes_\alpha \Gamma$ has finite nuclear dimension.

This was achieved as a further generalization of the previous slide. The resulting concept of Rokhlin dimension needed to be a lot more sophisticated than before, as well as the topological considerations leading to finite Rokhlin dimension of the action $\Gamma \curvearrowright \mathcal{C}(X)$.

Problem 1: One obtains upper bounds of the nuclear dimension, where one of the involved terms grows at least as fast as the asymptotic dimension of Γ. This means that the Rokhlin dimension approach is simply unsuitable to treat groups more complicated than the above, even the example $\Gamma = \mathbb{Z}^\infty$.

Problem 2: The proof suggests that the growth rate of Γ is relevant.

Definition

Let Γ be a group with a finite generating set $S = S^{-1}$. We can define a left-invariant metric on Γ via

$$d(g_1, g_2) = \min \left\{ n \geq 0 \mid g_1^{-1} g_2 = h_1 \cdots h_n,\ h_j \in S \right\}.$$

Although this metric depends on S, its large-scale behavior does not. We denote by $B_S(r) \subseteq \Gamma$ the ball around 1_Γ with radious $r > 0$.

Definition

A group Γ generated by a finite set $S = S^{-1}$ is said to have

- polynomial growth, if there exists some $\ell \geq 1$ and a constant $c > 0$ such that $|B_S(n)| \leq c n^\ell$ for all $n \geq 1$.
- subexponential growth, if $\lim_{n\to\infty} \frac{|B_S(n+1)|}{|B_S(n)|} = 1$.
- exponential growth, if otherwise.

Theorem (Gromov)

Among the finitely generated groups, those with polynomial growth are exactly the virtually nilpotent groups.

In other words, the Rokhlin dimension approach covers the case of polynomial growth, but not much beyond that.

Notation

Let us say that an arbitrary group Γ has local subexponential growth, if all its finitely generated subgroups have subexponential growth.

Theorem (Elliott–Niu, Niu)

For any free minimal action $\alpha : \mathbb{Z}^m \curvearrowright X$ *on any compact metric space with* $\mathrm{mdim}(\alpha) = 0$*, the crossed product* $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}^m$ *is* $\mathcal{Z}$*-stable.*

Mean dimension is a known invariant value in $[0, \infty]$ for topological dynamics due to Gromov and Lindenstraus–Weiss. Previous work by Giol–Kerr provides evidence towards the suspicion that the assumption $\mathrm{mdim}(\alpha) = 0$ may indeed be necessary for such a conclusion, so the above result is close to the optimal result for $\mathbb{Z}^m$-actions.

The proof of the above result is a very sophisticated implementation of Phillips' *large subalgebra* machinery, paired with some insights related to my work with Kerr that we are about to discuss. A similar result for other groups Γ was also proved by Niu, but nothing quite as final as the above.

It was perhaps first noticed by Winter that one can study a topological dynamical system $\Gamma \curvearrowright X$ for similar kinds of comparison properties as Cuntz semigroups of C^*-algebras.

Definition

Let $\alpha : \Gamma \curvearrowright X$ be an action of a countable group on a compact metric space. Suppose that $C \subseteq X$ is closed and $U \subseteq X$ is open. We say that C is dynamically subequivalent to U, written $C \prec U$, if we find pairwise disjoint open subsets $U_1, \dots, U_\ell \subseteq U$ and group elements $g_1, \dots, g_\ell \in \Gamma$ such that $C \subseteq \bigcup_{j=1}^{\ell} \alpha_{g_j}(U_j)$.

Definition

We say $\alpha : \Gamma \curvearrowright X$ has dynamical comparison, if the following is true. Whenever $C \subseteq X$ is closed and $U \subseteq X$ is open, one has the implication

$$\forall \mu \in M_\alpha(X) : \ \mu(C) < \mu(U) \implies C \prec U.$$

Definition

Let $\alpha : \Gamma \curvearrowright X$ be an action.

- A *tower* (w.r.t. α) is a pair (S, B) for a finite set $S \subseteq \Gamma$ and a set $B \subseteq X$ with the property that $\alpha_g(B) \cap \alpha_h(B) = \emptyset$ for all $g \neq h$ in S. We call S the shape of the tower. If B is open or clopen, we call the tower open or clopen.
- A *castle* (w.r.t. α) is a finite collection of towers $(S_j, B_j)_{j=1,\dots,\ell}$ such that $\alpha_g(B_j) \cap \alpha_h(B_i) = \emptyset$ for all $g \in S_j$, $h \in S_i$ with $i \neq j$. If the towers are open or clopen, we call the castle open or clopen.
- The local diameter of a castle $(S_j, B_j)_{j=1,\dots,\ell}$ is the maximal diameter of the sets $\alpha_g(B_j)$ for $j \leq \ell$ and $g \in S_j$.
- Let $\varepsilon > 0$ and $F \subseteq \Gamma$ a finite subset. We say a castle $(S_j, B_j)_{j=1,\dots,\ell}$ is (F, ε)-invariant, if all of its shapes are (F, ε)-invariant, i.e.

$$\max_{1 \leq j \leq \ell} \frac{\left|\bigcap_{g \in F} g S_j\right|}{|S_j|} \geq 1 - \varepsilon.$$

Definition (Kerr, generalizing a concept of Matui)

Let $\alpha : \Gamma \curvearrowright X$ be an action of an amenable group on a compact metric space. We say α is *almost finite*, if the following holds. For every $\varepsilon > 0$ and finite set $F \subseteq \Gamma$, there exists an (F, ε)-invariant open castle $(S_j, B_j)_{j=1,\dots,\ell}$ of diameter at most ε, and such that there exist subsets $S'_j \subset S_j$ with $|S'_j| < \varepsilon |S_j|$ for $j = 1, \dots, \ell$ such that

$$X \setminus \Big(\bigcup_{j=1}^{\ell} \bigcup_{g \in S_j} \alpha_g(B_j) \Big) \prec \bigcup_{j=1}^{\ell} \bigcup_{g \in S'_j} \alpha_g(B_j).$$

Stated like this, the property is reminiscent of the Ornstein–Weiss Rokhlin lemma for free measure-preserving actions of amenable groups, which is used extensively in ergodic theory. It characterizes freeness by saying that up to a small measure defect, the measure space can almost be exhausted by almost invariant measurable castles.

Theorem (Kerr)

Let $\alpha : \Gamma \curvearrowright X$ be a free minimal action of an amenable group. If α is almost finite, then $\mathcal{C}(X) \rtimes_\alpha \Gamma$ is $\mathcal{Z}$-stable.

Advantage: No specific structure of Γ is needed apart from amenability!

For those among you who happen to be familiar with techniques related to $\mathcal{Z}$-stability, here is a word on the core ideas: Every almost invariant castle with small diameter gives rise (in a non-trivial way) to an order zero map

$$\varphi : M_2 \xrightarrow{\perp} \mathcal{C}(X) \rtimes_\alpha \Gamma$$

whose range is approximately central. The "small remainder" condition translates to the element $\mathbf{1} - \varphi(\mathbf{1})$ being arbitrarily small in the Cuntz semigroup of the crossed product. In the end this allows one to show that $\mathcal{C}(X) \rtimes_\alpha \Gamma$ is *tracially $\mathcal{Z}$-stable* in the sense of Hirshberg–Orovitz, which implies $\mathcal{Z}$-stability by their theorem.

The above theorem motivated Kerr's slogan that almost finiteness is the dynamical analog of nuclearity in conjunction with $\mathcal{Z}$-stability!

Definition (Ad-hoc)

Let $\alpha : \Gamma \curvearrowright X$ be an action of an amenable group on a compact metric space. Let us say that a closed set $C \subseteq X$ is α-null, if $\mu(C) = 0$ for all $\mu \in M_\alpha(X)$.

Definition (Lindenstrauss)

Let $\alpha : \Gamma \curvearrowright X$ be an action of an amenable group on a compact metric space. It is said to have the *small boundary property* (SBP), if X has a basis of open sets $U \subseteq X$ for which ∂U is α-null.

Conjecturally, the SBP and the aforementioned condition "$\mathrm{mdim}(\alpha) = 0$" are equivalent for free actions of amenable groups. The SBP is also what actually drives the results of Elliott–Niu and Niu.

It is not so hard to prove that almost finiteness implies the SBP, but the reverse direction is a major open problem.

Definition (Kerr–S)

Let $\alpha : \Gamma \curvearrowright X$ be an action of an amenable group on a compact metric space. We say α is *almost finite in measure*, if the following holds. For every $\varepsilon > 0$ and finite set $F \subseteq \Gamma$, there exists an (F, ε)-invariant open castle $(S_j, B_j)_{j=1,\dots,\ell}$ of diameter at most ε, and such that

$$\forall \mu \in M_\alpha(X) : \mu\Big(X \setminus \Big(\bigcup_{j=1}^{\ell} \bigcup_{g \in S_j} \alpha_g(B_j)\Big)\Big) \leq \varepsilon.$$

Theorem (Kerr–S)

A free action $\alpha : \Gamma \curvearrowright X$ of an amenable group is almost finite in measure if and only if α has the SBP.

Corollary

A free action $\alpha : \Gamma \curvearrowright X$ of an amenable group is almost finite if and only if α has the SBP and has dynamical comparison.

Our characterization of the SBP in terms of almost finiteness in measure relies on the following new characterization of the SBP:

Theorem (Kerr–S)

Let $\alpha : \Gamma \curvearrowright X$ *be an arbitrary action. Then* α *has the SBP if and only if there exists an action* $\beta : \Gamma \curvearrowright Y$ *with* $\dim(Y) = 0$ *and an equivariant factor map*

$$\pi : (Y, \beta) \twoheadrightarrow (X, \alpha)$$

such that images of disjoint clopen sets in Y *under* π *intersect only as* α*-null sets in* X*.*

Our proof of almost finiteness in measure then involves two major steps. Firstly, we prove directly that all free Γ-actions on totally disconnected spaces are almost finiteness in measure, using techniques of Ornstein–Weiss. Secondly, given a free action $\alpha : \Gamma \curvearrowright X$ with the SBP, we use the properties of the above factor map π and the fact that β is almost finite in measure to deduce almost finiteness in measure for α.

We also use this fact to prove that transformation group C^*-algebras always satisfy the Toms–Winter conjecture.

Theorem (Castillejos–Evington–Tikuisis–White–Winter)

Let A be a separable simple nuclear unital C^*-algebra. If A has uniform property Gamma, then the Toms–Winter conjecture holds for A. In other words, A has finite nuclear dimension iff A is $\mathcal{Z}$-stable iff the Cuntz semigroup of A is almost unperforated.

Theorem (Kerr–S)

If $\alpha : \Gamma \curvearrowright X$ is almost finite in measure, then $\mathcal{C}(X) \rtimes_\alpha \Gamma$ has uniform property Gamma. In particular, if α is free and has the SBP, then the Toms–Winter conjecture holds for the transformation group C^*-algebra $\mathcal{C}(X) \rtimes_\alpha \Gamma$.

This theorem is used in the final step of the proof to get the desired conclusion in Niu's theorem about $\mathbb{Z}^m$-actions with mean dimension zero.

Theorem (Kerr–S)

Let Γ be a given amenable group. Suppose that every free action $\beta : \Gamma \curvearrowright Y$ with $\dim(Y) = 0$ is almost finite. Then it follows that every free action $\alpha : \Gamma \curvearrowright X$ with $\dim(X) < \infty$ is almost finite.

This observation relies on a similar but different extension trick compared to before. For every free action $\alpha : \Gamma \curvearrowright X$ with $\dim(X) < \infty$, we show that it is possible to find an action $\beta : \Gamma \curvearrowright Y$ with $\dim(Y) = 0$ and an equivariant factor map

$$\pi : (Y, \beta) \to (X, \alpha)$$

which has even stronger properties than stated before. This allows us to prove that dynamical comparison passes from β to α, which essentially proves the theorem.

As it turns out, it is possible to reconsider actions of groups with polynomial growth in light of the previous slide. With a rather direct counting argument, we could provide a quick proof of the following:

Proposition

Let Γ be a finitely generated group with polynomial growth. Then every action $\beta : \Gamma \curvearrowright Y$ with $\dim(Y) = 0$ has dynamical comparison.

As a direct corollary, we can recover the previous theorems that were proved with Rokhlin dimension techniques:

Corollary

Let Γ be a finitely generated group with polynomial growth. Then every free action $\alpha : \Gamma \curvearrowright X$ with $\dim(X) < \infty$ is almost finite. Consequently, if α is also minimal, then the transformation group C^*-algebra $\mathcal{C}(X) \rtimes_\alpha \Gamma$ is $\mathcal{Z}$-stable and classifiable.

While we were working out this theory, we were happy to be notified about the following result obtained by people working in topological dynamics at the same time.

Theorem (Downarowicz–Zhang)

Let Γ be a group with local subexponential growth. Then every action $\beta : \Gamma \curvearrowright Y$ with $\dim(Y) = 0$ has dynamical comparison.

The proof of this result consists of hard combinatorics that exploits the subexponential growth in a sophisticated way. Freeness of the action is notably not needed here.

As a consequence, we obtain the main result of our work.

Theorem (Kerr–S)

Let Γ be a group with local subexponential growth. Then every free action $\alpha : \Gamma \curvearrowright X$ with $\dim(X) < \infty$ is almost finite. Consequently, if α is also minimal, then the transformation group C^*-algebra $\mathcal{C}(X) \rtimes_\alpha \Gamma$ is $\mathcal{Z}$-stable and classifiable.

Within the last month, Conley–Jackson–Marks–Seward–Tucker-Drob have extended the main result from the previous slide to actions of certain elementary amenable groups that can have genuine exponential growth.

The case of general amenable groups still remains open.

Question

Is it true that all actions of amenable groups $\Gamma \curvearrowright Y$ with $\dim(Y) = 0$ have dynamical comparison?

As the work of Downarowicz–Jhang shows, this question can be rephrased entirely as a complicated combinatorial question involving certain Følner sets in Γ. It is thus unclear whether C^*-algebraic techniques can be of any help to solve this remaining piece of the puzzle.

Thank you for your attention!