The Realization Problem for von Neumann regular rings

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Abend Seminars

18 June 2020

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$\mathcal{V}(R)$ is conical $(x + y = 0 \Rightarrow x = y = 0)$, and has an order unit $u = [R]$.

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a	x	y
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1rst THOUGHT : All conical refinement monoids arise as $\mathcal{V}$-monoids of regular rings, but..

Counterexample (Wehrung '98)

Build a monoid of size $\aleph_2$ that can not be realized by a regular ring.

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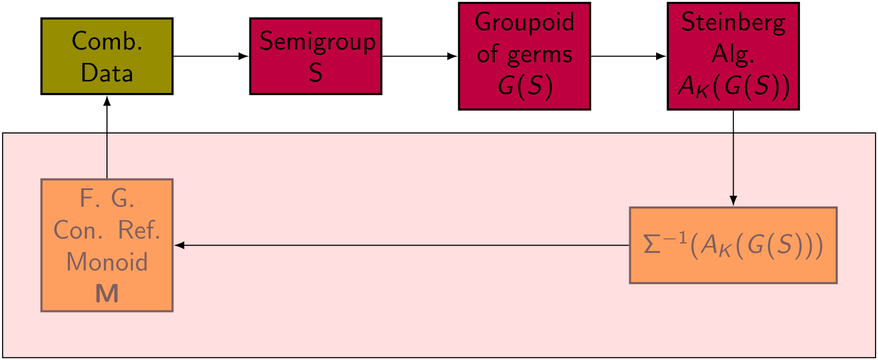
Theorem (Ara-B-Pardo '20)

Every f. g. conical refinement monoid M is realizable by a regular ring R, i.e.

$$\mathcal{V}(R) \cong M.$$

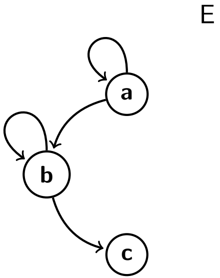
Strategy



References:

1. P. Ara, J. Bosa E. Pardo "Refinement monoids and adaptable separated graphs," Semigroup Forum, 10.1007/s00233-019-10077-2
2. P. Ara, J. Bosa, E. Pardo, A. Sims, "The Groupoids of Adaptable Separated graphs and Their Type semigroups." IMRN, 10.1093/imrn/rnaa022
3. P. Ara, J. Bosa, E. Pardo, "The realization problem for finitely generated refinement monoids", Selecta Mathematica 26 (2020), no.3, 33.

Similar strategy was done for monoids arising from graphs and using Leavitt path Algebras:



$$M(E) = \{a, b, c \mid a = a + b, b = b + c\}$$

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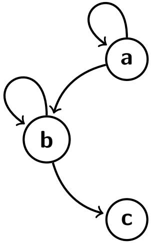
E



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Theorem (Ara-Brustenga'09, Ara-Pardo-Moreno '07)

All conical monoids arising from finite graphs E are realizable by regular rings, in particular

$$M(E) \cong \mathcal{V}(L_K(E))$$

for any field K.

Example (Ara-Pardo-Wehrung)

The conical and refinement monoid $M := \{p, a, b \mid p = p + a = p + b\}$ is not a graph monoid.

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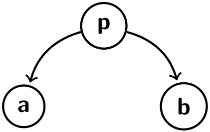
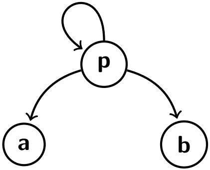
E_1



$M(E_1) = \{p, a, b \mid p = a + b\}$

E_2



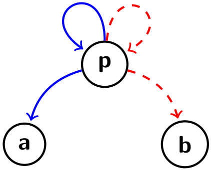
$M(E_2) = \{p, a, b \mid p = p + a + b\}$

NONE of those works

Example (Ara-Pardo-Wehrung)

The conical and refinement monoid $M := \{p, a, b \mid p = p + a = p + b\}$ is not a graph monoid.

(E, C)

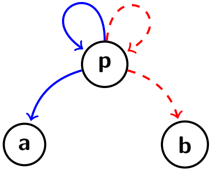


$$M(E, C) = \{p, a, b \mid p = p + a = p + b\}$$

Separated graphs (E, C) were introduced by Ara-Goodearl (2012) as a pair (E, C), where E is a direct graph and C is a partition of the set of edges of E. But $M(E, C)$ is **not** a refinement monoid in general.

Based on Ara-Pardo (Israel J. Math. '16), where they characterize the combinatorial data of all f.g. conical refinement monoids, and looking at

(E, C)



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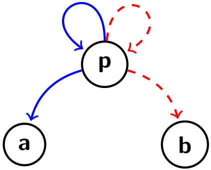

$$M(E, C) = \{p, a, b \mid p = p + a = p + b\}$$

Theorem (Ara-B-Pardo '19)

For any finitely generated conical refinement monoid M, there exists an adaptable separated graph (E, C) such that $M \cong M(E, C)$.

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Definition

Let (E, C) be a finitely separated graph and let $(I, \leq)$ be the poset arising from antisymmetrization of $(E^0, \leq)$. It is **adaptable** if $I = I_{free} \sqcup I_{reg}$ is finite , and a family of subgraphs $\{E_p\}_{p \in I}$ of E such that:

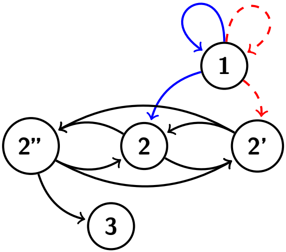
1. $E^0 = \bigsqcup_{p \in I} E_p^0$, where E_p is a transitive row-finite graph if $p \in I_{reg}$ and $E_p^0 = \{v^p\}$ is a single vertex if $p \in I_{free}$.
2. For $p \in I_{reg}$ and $w \in E_p^0$, we have that $|C_w| = 1$ and $|s_{E_p}^{-1}(w)| \geq 2$. Moreover, all edges departing from w either belong to the graph E_p or connect w to a vertex $u \in E_q^0$, with $q < p$ in I.
3. For $p \in I_{free}$and not minimal, there exists $k(p)$ colours $C_{v^p} = \{X_1^{(p)}, \dots, X_{k(p)}^{(p)}\}$, and each
$$X_i^{(p)} = \{\alpha(p, i), \beta(p, i, 1), \beta(p, i, 2), \dots, \beta(p, i, g(p, i))\},$$
where $\alpha(p, i)$ is a loop,and $r(\beta(p, i, t)) \in E_q^0$ for $q < p$ in I.

Definition (adaptable Separated Graph)

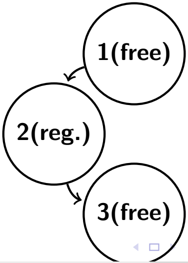
$(E, C, (I, \leq))$ with $I = I_{free} \sqcup I_{reg}$, and a family of $\{E_p\}_{p \in I}$ of E such that:

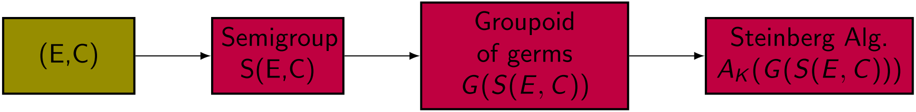
1. $E^0 = \bigsqcup_{p \in I} E_p^0$, where E_p is a transitive row-finite graph if $p \in I_{reg}$ and $E_p^0 = \{v^p\}$ is a single vertex if $p \in I_{free}$.
2. For $p \in I_{reg}$ and $w \in E_p^0$, we have that $|C_w| = 1$ and $|s_{E_p}^{-1}(w)| \geq 2$. All edges departing from w either are in E_p or connect w to $u \in E_q^0$ $(q < p)$.
3. If p free and not minimal, then it has $k(p)$ colors such that each color $X_i^{(p)} = \{\alpha(p,i), \beta(p,i,1), \beta(p,i,2), \dots, \beta(p,i,g(p,i))\}$.

(E, C)



Poset $(I, \leq)$

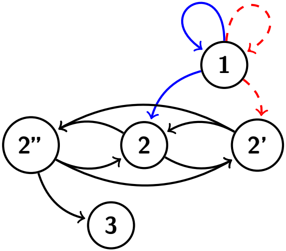




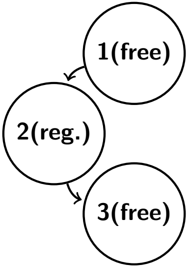
P. Ara, J. Bosa, E. Pardo, A. Sims, "The Groupoids of Adaptable Separated graphs and Their Type semigroups." IMRN, 10.1093/imrn/rnaa022

Given a adaptable separated graph (E, C), we define a "natural" inverse semigroup $S(E, C)$ build upon finite paths that arise from the separated graph.

(E, C)



Poset $(I, \leq)$



These are the concatenation of a "c-paths" and monomials at the components of the poset $(I, \leq)$. They are triples, for instances:

$$(\gamma, m(\mathbf{2}), \eta^*) \text{ satisfying } r(\gamma) = s(m(\mathbf{2})) \text{ and } r(m(\mathbf{2})) = r(\eta).$$

Remark

We introduce a set of auxiliary variables to each vertex to tame the natural relations associated to (E, C), without altering the associated monoid.

We study the semilattice of **idempotents** $\mathcal{E}$ in $S(E, C)$(described solely in terms of paths and monomials in the (E, C)), and the set of its infinite paths and characterize its tight filters.

Proposition

Given an adaptable separarted graph, there is a bijection between the set of infinite paths in $S(E, C)$ and the set of tight filters.

Given an adaptable separated graph, we build the groupoid $\mathcal{G}_{tight}(S(E,C))$ of germs of the canonical action of $S(E,C)$ on $\hat{\mathcal{E}}_{tight}$.Then, we characterize the associated Steinberg Algebra and C*-algebra.

Proposition

Let (E,C) be an adaptable separated graph. Then, the groupoid $\mathcal{G}_{tight}(S(E,C))$ is amenable and:

- *The Steinberg Algebra $A_K(\mathcal{G}_{tight}(S(E,C)))$ is isomorphic to $\mathcal{S}_K(E,C)$, the K-span of the elements of the inverse semigroup $S(E,C)$.*
- $C^*(S(E,C)) \cong C^*(\mathcal{G}_{tight}(S(E,C)))$.

We finish the notes (and the talk), speaking about the **Type semigroup** associated to a groupoid.

$\mathrm{Typ}(\mathcal{G})$, for any ample groupoid $\mathcal{G}$, is a new invariant that characterizes part of the structure theory of the associated reduced groupoid C*-algebra.

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Theorem (Ara-B-Pardo-Sims '19)

Let (E, C) be an adaptable separated graph, then

$$M(E, C) \cong \mathrm{Typ}(\mathcal{G}_{tight}(S(E, C))).$$

Corollary (Ara-B-Pardo-Sims '19)

For any finitely generated conical refinement monoid M, there exists an amenable groupoid $\mathcal{G}_{tight}(S(E, C))$, associated to an adaptable separated graph, such that

$$M \cong \mathrm{Typ}(\mathcal{G}_{tight}(S(E, C)))$$

Thanks for your attention !