Amenability for actions of groups on C*-algebras

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Abstract

Amenability for actions of groups on C*-algebras

In this lecture I will explain recent developments in the theory of amenability for actions of groups on C*-algebras and Fell bundles, based on joint works with Siegfried Echterhoff, Rufus Willett, Fernando Abadie and Damián Ferraro.

Our main results prove that essentially all known notions of amenability are equivalent. We also extend Matsumura's theorem to actions of exact locally compact groups on commutative C*-algebras and give a counter-example for the weak containment problem for actions on noncommutative C*-algebras.

Amenable groups

Let G be a locally compact group.

Recall: G is amenable if there exists a G-invariant state (an "invariant mean")

$$\varphi \colon L^\infty(G) \to \mathbb{C}.$$

Here $L^\infty(G)$ is endowed with the (left) translation G-action:

$$\lambda_t(f)(s) := f(t^{-1}s), \quad f \in L^\infty(G), s, t \in G.$$

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Characterizing amenability of groups

There are many equivalent ways to characterise amenability of G:

Proposition

TFAE:

(i) G *is amenable;*

(ii) *("almost invariant vectors") there is a net* $(\xi_i) \subseteq C_c(G) \subseteq L^2(G)$ *with* $\|\xi_i\|_2 \leq 1$ *and*

$$\langle \xi_i \,|\, \lambda_t(\xi_i) \rangle_2 \to 1 \quad \forall t \in G;$$

(iii) *(*$\lambda_G \preceq 1_G$*) there is a net of compactly supported continuous positive type functions* $\theta_i \colon G \to \mathbb{C}$ *such that* $\theta_i(t) \to 1$*;*

(iv) $C^*(G) = C^*_r(G)$*.*

Moreover, if G *is discrete, the above are equivalent to*

(v) $C^*_r(G)$ *is nuclear, or equiv.,* $W^*_r(G) = C^*_r(G)''$ *is injective.*

Amenable actions on spaces

Definition (Anantharaman-Delaroche 2000)

A (continuous) action of a l.c. group G on a l.c. space X is *amenable* if there is a net of (weak*-)continuous maps $m_i \colon X \to P(G)_1^+$ (probability measures), $x \mapsto m_i^x$ such that

$$\|m_i^{t\cdot x} - t \cdot m_i^x\| \to 0$$

uniformly for (t, x) in compact subsets of $G \times X$.

Example

(1) Proper actions, like the translation G-action on itself are always amenable.

(2) The action of a free group (or more generally a hyperbolic group) on its (Gromov) boundary is amenable.

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Exact locally compact groups

Theorem (Ozawa 2000, Brodzki-Cave-Li 2017)

A locally compact group G is exact iff it admits an amenable action on a compact space iff G acts amenably on the spectrum $\partial_u(G)$ of the C^*-algebra $C_{ub}(G)$ of uniformly continuous functions endowed with the (left) translation G-action.

In particular, a discrete group G is exact iff it acts amenably on $\beta(G) = \widehat{\ell^\infty(G)}$ iff $C_r^*(G)$ is exact.

Example

(1) Free groups and, more generally, hyperbolic groups are exact.

(2) All almost connected l.c. groups are exact.

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Moving towards noncommutative C*-algebras

Now assume that G acts (continuously) on a C*-algebra A via $\alpha\colon G \to \mathrm{Aut}(A)$. We can then assign to this two C^*-algebras:

the full crossed product $\rightsquigarrow A \rtimes_\alpha G$

the reduced crossed product $\rightsquigarrow A \rtimes_{\alpha,r} G$

Both can be viewed as completions of the convolution *-algebra $A \rtimes_{\alpha,\mathrm{alg}} G = C_c(G, A)$. One fundamental question is then:

When is

$$A \rtimes_\alpha G = A \rtimes_{\alpha,r} G?$$

$\rightsquigarrow$ weak containment property (WC)

This should be related to some sort of 'amenability' of the action.

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von Neumann amenable actions

When we allow a coefficient C*-algebra A things change/complicate! But at least for actions on W*-algebras the notion of amenability is similar:

Definition (Anantharaman-Delaroche 1979)

Let M be a W*-algebra (i.e. von Neumann algebra) endowed with a W*-continuous G-action γ by automorphisms. We say that (M, γ) is (von Neumann) amenable if there exists a G-equivariant projection

$$P\colon L^\infty(G, M) \to M.$$

Here $L^\infty(G, M) = L^\infty(G)\bar{\otimes}M$ is endowed with the diagonal G-action $\tilde{\gamma} = \lambda \otimes \gamma$ and we embed $M \hookrightarrow L^\infty(G, M)$ by constant functions.

Theorem (Anantharaman-Delaroche 1987)

(M, γ) is amenable iff $(Z(M), \gamma|)$ is amenable

Assume G is discrete. Then (M, γ) is amenable iff there is a net $(\xi_i) \subseteq C_c(G, Z(M)) \subseteq \ell^2(G, M)$ with $\|\xi_i\|_2 \leq 1$ for all i and

$$\langle \xi_i \,|\, \tilde{\gamma}_t(\xi_i) \rangle_2 \xrightarrow{weak*} 1 \quad \forall t \in G$$

where $\langle \xi \,|\, \eta \rangle_2 = \sum_{t \in G} \xi(t)^* \eta(t)$ denotes the M-valued inner product on $\ell^2(G, M) = \ell^2(G) \otimes M$.

iff there is a net of γ-positive type functions $(\theta_i) \subseteq C_c(G, Z(M))$ with

$$\theta_i(t) \xrightarrow{weak*} 1.$$

Moreover, if M is injective, then (M, γ) is amenable iff the W^*-crossed product $M \bar{\rtimes} G$ is injective.

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Moving towards C*-actions

Question: What for actions on C^*-algebras?

For discrete groups, Claire introduced:

Definition (Anantharaman-Delaroche 1987)

Let (A, α) be a C^*-action. We say that (A, α) is amenable if the enveloping (double dual) W*-algebra A^{**} with induced G-action α^{**} is amenable (as a W*-action).

In other words, (A, α) is amenable iff there is a G-equivariant projection $\ell^\infty(G, A^{**}) \to A^{**}$, iff there is a net $\xi_i \colon G \to Z(A^{**})$ of finitely supported functions with $\|\xi_i\|_2 \leq 1$ for all i and

$$\theta_i(t) := \langle \xi_i \,|\, \tilde{\alpha}_t^{**}(\xi_i) \rangle_2 \to 1 \quad \forall t \in G.$$

with respect to the w*-topology on A^{**}.

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Theorem (Anantharaman-Delaroche, 1987)

Let G be a discrete group. If (A, α) is amenable, then

$$A \rtimes_\alpha G = A \rtimes_{\alpha,r} G.$$

Moreover, if A is nuclear, then (A, α) is amenable iff $A \rtimes_{\alpha,r} G$ is nuclear.

Open standing question:

$$A \rtimes_\alpha G = A \rtimes_{\alpha,r} G \quad (WC) \quad \Rightarrow \quad (A, \alpha) \quad \text{amenable?}$$

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Here is a partial answer:

Theorem (Matsumura, 2014)

Let A be a unital, nuclear C^*-algebra with an action α of a discrete group G.

If A is commutative and G is exact, then (A, α) is amenable iff it has the (WC), i.e.,

$$A \rtimes_{\alpha} G = A \rtimes_{\alpha,r} G.$$

If G is exact and A is possibly noncommutative, then (A, α) is amenable iff $\alpha \otimes \alpha^{\mathrm{op}}$ has the (WC):

$$(A \otimes A^{\mathrm{op}}) \rtimes_{\alpha \otimes \alpha^{\mathrm{op}}} G = (A \otimes A^{\mathrm{op}}) \rtimes_{\alpha \otimes \alpha^{\mathrm{op}},r} G.$$

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Question

What about locally compact groups?

Main problem: the induced G-action α^{**} on A^{**} need not be pointwise weak*-continuous, that is, A^{**} is a not a G-W*-algebra in general.

Example

The problem already happens for example for $A = C_0(G)$. In this case A^{**} is too big, contains all measurable functions on G and A^* contains all evaluation functionals (dirac measures).

Theorem (Ikunishi 1988, B-Echterhoff-Willett 2020)

For a G-C^*-algebra (A, α), there exists a (unique) G-W-algebra (A''_α, α'') containing A as a weakly dense G-algebra in such a way that every G-equivariant *-homomorphism $\pi \colon A \to M$ into a G-W*-algebra M extends to a normal G-equivariant *-homomorphism $A''_\alpha \to M$.*

Ikunishi's construction: consider the continuous part A^*_c of the dual A^* and realise A''_α as the dual of A^*_c.

Our construction: Define A''_α as the image of A^{**} inside $(A \rtimes_\alpha G)^{**}$ or, equivalently, in $(A \rtimes_{\alpha,r} G)^{**}$.

Remark

If G is discrete, then $A''_\alpha = A^{**}$: in this case A embeds into $A \rtimes G$ and therefore A^{**} embeds into $(A \rtimes_\alpha G)^{**}$ or $(A \rtimes_{\alpha,r} G)^{**}$.

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Example

Consider $A = C_0(G)$ with the translation G-action. Then $A_c^* = L^1(G)$ and therefore $A''_\alpha = L^\infty(G)$.

This can also be seen from the crossed-product picture:

$$C_0(G) \rtimes G \cong \mathbb{K}(L^2 G)$$

and the image of $C_0(G)^{**}$ into $(C_0(G) \rtimes G)^{**} \cong \mathbb{B}(L^2 G)$ is

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Definition (BEW 2020)

We say that a G-C^*-algebra (A, α) is amenable if there is a net $(\theta_i) \subseteq C_c(G, Z(A''_\alpha)_c)$ of positive type functions with $\|\theta_i(e)\| \leq 1$ for all i and

$$\theta_i(t) \to 1 \quad \text{(ultra)weakly and uniformly on compacts}$$

iff there is a net (ξ_i) in the unit ball of $L^2(G, Z(A''_\alpha)_c)$ with

$$\theta_i(t) = \langle \xi_i \,|\, \alpha''_t(\xi_i) \rangle_2 \to 1 \quad \text{(ultra)weakly and uniformly on compacts.}$$

Remark

Here $Z(A''_\alpha)_c$ denotes the continuous part w.r.t. α''.

Definition

We also say that (A, α) is strongly amenable if there is a net $(\theta_i) \subseteq C_c(G, Z\mathcal{M}(A)_c)$ of positive type functions with $\|\theta_i(e)\| \leq 1$ for all i and

$$\theta_i(t) \to 1 \quad \text{strictly and uniformly on compacts.}$$

Remark

It follows more or less from the definition that a commutative G-C^*-algebra $(C_0(X), \alpha)$ is strongly amenable iff the underlying G-action on X is (topologically) amenable.

Theorem (BEW 2020)

If G is exact, TFAE for a G-C^*-algebra (A, α):

- (A, α) *is amenable;*
- (A''_α, α'') *is vN-amenable in the sense of Delaroche, i.e., there is G-equivariant projection $L^\infty(G, A''_\alpha) \to A''_\alpha$);*
- $(A''_{\alpha,c}, \alpha''|)$ *is strongly amenable, i.e., the G-action on the spectrum of $Z(A''_\alpha)_c$ is topologically amenable;*
- *there is a G-equivariant ucp map $L^\infty(G) \to Z(A''_\alpha)$;*
- *there is a G-equivariant ucp mp $C_{ub}(G) \to Z(A''_\alpha)_c$.*
- *(extension of Matsumura's theorem) the diagonal G-action $\alpha \otimes \alpha^{\mathrm{op}}$ on $A \otimes_{\max} A^{\mathrm{op}}$ has the WC:*

$$(A \otimes_{\max} A^{\mathrm{op}}) \rtimes_{\alpha\otimes\alpha^{\mathrm{op}}} G = (A \otimes_{\max} A^{\mathrm{op}}) \rtimes_{\alpha\otimes\alpha^{\mathrm{op}},r} G.$$

Proof idea of some implications:

For example, if there is a G-equivariant projection $L^\infty(G, A''_\alpha) \to A''_\alpha$ we restrict it to centers and compose with the canonical embedding $L^\infty(G) \to L^\infty(G, Z(A''_\alpha))$ to get a ucp G-map

$$L^\infty(G) \to Z(A''_\alpha)$$

Further restricting to continuous parts we then get a ucp G-map

$$C_{ub}(G) \to Z(A''_\alpha)_c$$

Using now exactness of G, i.e., (SA) of $C_{ub}(G)$, we get (SA) for $Z(A''_\alpha)_c$.

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Remark

(1) In the above form, exactness of G cannot be removed because the existence of a topologically amenable G-action on a compact space characterises exactness of G (by Ozawa & Brodzki-Cave-Li).

(2) Also, recent examples of Suzuki (published in 2019) show that there are simple G-C^*-algebras of exact groups that are amenable but not strongly amenable.

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(2) Also, recent examples of Suzuki (published in 2019) show that there are simple G-C^*-algebras of exact groups that are amenable but not strongly amenable.

Commutative case

Theorem (BEW 2020 - Matsumura's theorem extended)

If an exact l.c. group G acts on a l.c. space X, then amenability of the induced G-action on $A = C_0(X)$ is equivalent to the WC:

$$C_0(X) \rtimes_\alpha G = C_0(X) \rtimes_{\alpha,r} G.$$

And if G and X are 2^{nd} countable, then this is further equivalent to measurewise amenability of X in the sense of Delaroche-Renault:

for every quasi-invariant measure μ on X, the G-W-algebra $L^\infty(X,\mu)$ with the induced G-action is vN-amenable.*

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for every quasi-invariant measure μ on X, the G-W-algebra $L^\infty(X,\mu)$ with the induced G-action is vN-amenable.*

Question

What happens if G is not exact? The main ingredient in the exact case is the result of Ozawa & Brodzki-Cave-Li showing that the action on $C_{ub}(G)$ is strongly amenable iff G is exact.

If G is not exact, there is no unital G-C^*-algebra carrying a (strongly) amenable action. So amenability is a bit misterious in the non-exact case.

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Nevertheless, just after we posted our preprint in the arxiv, the following has been shown:

Theorem (Bearden-Crann 2020)

Let G be any locally compact group. TFAE for a G-C^*-algebra:

- (A, α) *is amenable;*
- (A''_α, α'') *is vN-amenable (a la Delaroche).*

Moreover, if $A = C_0(X)$ is commutative, all this is equivalent to

- (A, α) *is strongly amenable, i.e., the G-action on X is topologically amenable.*

Corollary

If G is exact, then $C_0(X) \rtimes_\alpha G = C_0(X) \rtimes_{\alpha,r} G$ iff the G-space X is topologically amenable.

Characterizing weak containment

For non-exact G we have another related result:

Theorem (B.,Echterhoff-Willett, 2020)

If $A = C_0(X)$ is commutative, then $A \rtimes_\alpha G = A \rtimes_{\mathrm{inj}} G$ iff A has the (continuous) G-WEP:

By definition, this means that every G-embedding $A \hookrightarrow B$ admits a G-equivariant ccp splitting $B \to A''_\alpha$.

Here $A \rtimes_{\mathrm{inj}} G$ denotes the maximal injective crossed product: $A \rtimes_{\mathrm{inj}} G$ is the completion of $A \rtimes_{\mathrm{alg}} G = C_c(G, A)$ with respect to

$$\|f\| = \inf\{\|\iota \circ f\|_{B \rtimes_\alpha G} : \text{ for } \iota : A \hookrightarrow B \text{ a } G\text{-embedding}\}.$$

Characterizing weak containment

Theorem (B.,Echterhoff-Willett, 2020)

For a general G-action (A, α) we have

$$A \rtimes_{\alpha} G = A \rtimes_{\mathrm{inj}} G$$

iff every covariant representation $(\pi, U) \to (A, G) \to \mathbb{B}(H)$ is G-injective in the sense that every embedding $\iota \colon A \hookrightarrow B$ admits a G-equivariant ccp map $\varphi \colon B \to \mathbb{B}(H)$ with $\varphi \circ \iota = \pi$.

Remark: If G is exact, then $A \rtimes_{\mathrm{inj}} G = A \rtimes_{\alpha,r} G$, so that the above result gives a characterisation for $A \rtimes_{\alpha} G = A \rtimes_{\alpha,r} G$.

Proof ideas:

$A \rtimes_\alpha G = A \rtimes_{\text{inj}} G$ means that every G-embedding $A \hookrightarrow B$ extends to an embedding $A \rtimes_\alpha G \hookrightarrow B \rtimes_\beta G$.

Now we use Arverson's theorem to extend the representation $\pi \rtimes U \colon A \rtimes_\alpha G \to \mathbb{B}(\mathcal{H})$ to a ccp map $\mathcal{M}(B \rtimes_\beta G) \to \mathbb{B}(\mathcal{H})$ and then 'restrict' to B to get the desired ccp G-map $B \to \mathbb{B}(\mathcal{H})$.

In the commutative case we use Haargerup's standard representation of A''_α, where this commutative vN-algebra sits as a masa, that is, $\pi(A)' = \pi(A)'' \cong A''_\alpha$. Now use a multiplicative domain argument to prove that the ccp G-map $\varphi \colon B \to \mathbb{B}(\mathcal{H})$ has image inside $\pi(A)'$.

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Characterizing weak containment

Based on this we could also prove the following:

Theorem (B.,Echterhoff-Willett, 2020)

If G is exact, then $A \rtimes_\alpha G = A \rtimes_{\alpha,r} G$ iff (A,α) is commutant amenable in the sense that every covariant representation $(\pi, U)\colon (A,G) \to \mathbb{B}(H)$ admits a net $\xi_i\colon G \to \pi(A)'$ of compactly supported continuous functions with $\|\xi_i\|_2 \le 1$ for all i and

$$\langle \xi_i \,|\, \tilde{\beta}_t(\xi_i) \rangle_2 \to 1 \quad \forall t$$

with respect to the (ultra)weak topology.

Here we endow $\mathbb{B}(H)$ and also $\pi(A)'$ with the action $\beta_t = \mathrm{Ad}_{U_t}$ by conjugation with U.

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Proof idea:

The main point is the observation that for a G-injective covariant representation (π, U) and any unital G-C^*-algebra C, we get a ucp G-map $C \to \pi(A)'$. For this we just use the canonical G-embedding

$$A \hookrightarrow \mathcal{M}(A \otimes C).$$

Now apply this to $C = C_{ub}(G)$ and use that G is exact.

Amenability always implies commutant amenability.

Question

Does commutant amenability imply amenability?

Theorem (BEW 2020)

There are exact locally compact groups (like $G = PSL_2(\mathbb{R})$) acting on $\mathbb{K}$ and satisfying the WC (and hence commutant amenable):

$$\mathbb{K} \rtimes G = \mathbb{K} \rtimes_r G$$

Such actions are never amenable unless G is amenable.

Idea: find a 2-cocycle ω on G such that $C^*(G, \omega) = C^*_r(G, \omega)$.

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Relation to approximation properties

The following is an alternative to amenability that avoids centrality, commutants and going to biduals:

Definition (Exel 1997, Exel-Ng 2002)

A G-C^*-algebra (A, α) has the approximation property (AP) if there is a bounded net $(\xi_i) \subseteq C_c(G, A) \subseteq L^2(G, A)$ such that

$$\langle \xi_i \,|\, a\tilde{\alpha}_t(\xi_i)\rangle_2 \to a$$

in norm for all $a \in A$ and $t \in G$ (uniformly on compacts).

Remark

The above extends naturally to Fell bundles over groups – in particular partial actions – and was indeed originally introduced in this setting.

In joint work with Abadie-Ferraro we also introduced a weak version of the AP called the wAP (the norm convergence is replaced by weak convergence) and proved:

Theorem (Abadie-B-Ferraro 2019)

For actions of discrete groups, or more generally Fell bundles,

$$(AP) \Longrightarrow (wAP) \Longleftrightarrow (A)$$

Moreover, all these notions are invariant under (weak) equivalences of Fell bundles. In particular a Fell bundle $\mathcal{B} = (B_t)_{t\in G}$ over a discrete group G has the (AP) iff the corresponding dual G-action on $k(\mathcal{B}) := C^*(\mathcal{B}) \rtimes G$ has the AP.

For Fell bundles whose unit fibre C^*-algebra $\mathcal{B}_e$ is Morita equivalent to a commutative C^*-algebra, we have

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For Fell bundles whose unit fibre C^**-algebra* $\mathcal{B}_e$ *is Morita equivalent to a commutative* C^**-algebra, we have*

$$(AP) \Longleftrightarrow (wAP).$$

We introduce yet another variant of the AP:

Definition

A G-C^*-algebra (A, α) has the quasi-central approximation property (QAP) if there is a net $(\xi_i) \subseteq C_c(G, A)$ with $\|\xi_i\|_2 \leq 1$ for all i,

- $\langle \xi_i \,|\, \tilde{\alpha}_t(\xi_i) \rangle \to 1$ strictly in $\mathcal{M}(A)$; and
- $\|\xi_i \cdot a - a \cdot \xi_i\|_2 \to 0$ for all $a \in A$.

Remark

(1) It is easy to see that $(SA) \Rightarrow (QAP) \Rightarrow (AP)$.

(2) QAP is closed under direct limits. Suzuki examples of nuclear simple crossed products are direct limits of SA-actions, so they have the QAP (but are not SA).

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It turns out that all these notions are equivalent:

Theorem (BEW 2020)

For a discrete group G and a G-C^*-algebra (A, α), TFAE:

- (A, α) *is amenable*
- (A, α) *has the QAP*
- (A, α) *has the AP*

Remark

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Using all this we could answer the following question of Exel:

Corollary

If the (reduced) cross-sectional C^*-algebra $C^*_r(\mathcal{B})$ of a Fell bundle $\mathcal{B} = (B_t)_{t \in G}$ over a discrete group G is nuclear, then $\mathcal{B}$ has the AP.

Matsumura's theorem for Fell bundles?

The following is, however, still open:

Question

For a Fell bundle $\mathcal{B}$ over a discrete group G, does

$$C^*(\mathcal{B}) = C^*_r(\mathcal{B})$$

imply that $\mathcal{B}$ has the AP?

We have a positive answer for Fell bundles associated with partial actions on spaces – joint work in progress with Ferraro-Sehnem, but even the simpler form of this question is open:

Question

For a 2-cocycle ω on a discrete group G, does

$$C^*(G,\omega) = C^*_r(G,\omega) \Longrightarrow G \textit{ is amenable?}$$