An Irrational-slope Thompson's Group

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Introduction

History

In 1969, in a context of logic, Richard J. Thompson introduced a family of groups (called since Thompson's groups) which captured properties of commutativity and associativity.

Higman realised these groups were important from the group-theoretic point of view. In particular, the group originally called *Thompson's group* (now called V) was the first known example of a finitely presented, infinite, simple group.

History

An interesting subgroup of V, called F, was described by Geoghegan in 1979, and he conjectured that:

1. F has infinite cohomological dimension.
2. F is simply connected at infinity.
3. F has no nonabelian free subgroups.
4. F is not amenable.

The first three properties were readily proved by several people (Brown, Geoghegan, Brin, Squier), and property 4 is famously still open.

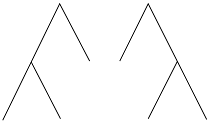
Thompson's group F

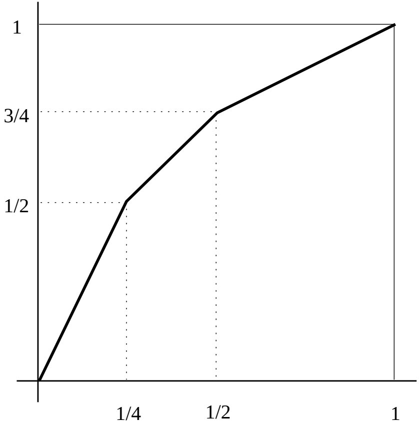
Thompson's group F is the group of homeomorphisms of $[0, 1]$ such that:

- are piecewise linear and orientation-preserving,
- have breakpoints in $\mathbb{Z}[\frac{1}{2}]$,
- slopes are powers of 2.

Thompson's group F

Elements can be obtained with subdivisions of the interval, which can be encoded with carets:





Generalisations

Several authors generalised this group to other breaks and slopes, for instance $\mathbb{Z}[\frac{1}{n}]$ and powers of n, or allowing powers of 2 and 3 and breaks of the type $a/2^n3^m$.

Bieri and Strebel wrote a wonderful memoir about groups of PL maps. Given $A \subset \mathbb{R}$ a subring, and given $\Lambda \subset A^*$ a subgroup of units of A, they define the group $G(I, A, \Lambda)$, which is the group of orientation-preserving, piecewise-linear homeomorphisms of I with breaks in A and slopes in Λ.

Hence F is the group $G([0,1], \mathbb{Z}[\frac{1}{2}], \langle 2 \rangle)$.

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Definition

Definition

Let $\tau = \frac{\sqrt{5}-1}{2} = 0.6180339887...$, which is a zero of the polynomial $X^2 + X - 1$.

Consider the ring $\mathbb{Z}[\tau]$ of elements of the type $a + b\tau$, where a and b are integers.

Then the group F_τ is the group $G([0,1], \mathbb{Z}[\tau], \langle\tau\rangle)$ in Bieri-Strebel notation, that is, the group of piecewise-linear maps of the interval $[0,1]$ with breakpoints in $\mathbb{Z}[\tau]$ and slopes powers of τ.

Subdivisions

Since $1 = \tau + \tau^2$, the unit interval can be split into two intervals of lengths τ and τ^2, which can be done in two ways:

$$[0, 1] = [0, \tau] \cup [\tau, 1] \qquad [0, 1] = [0, \tau^2] \cup [\tau^2, 1].$$

And any smaller interval can be subdivided further using $\tau^k = \tau^{k+1} + \tau^{k+2}$ (for all $k = 0, 1, 2, \ldots$).

Then, the interval can be subdivided into n intervals whose lengths are all powers of τ.

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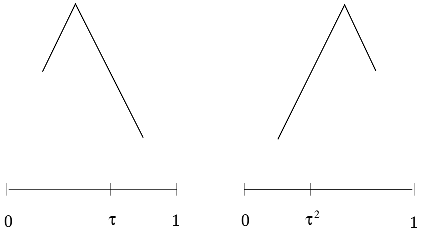
Subdivisions

Given two such subdivisions in n intervals, an element of F_τ can be obtained by mapping the intervals linearly in order-preserving fashion.

The group F_τ was introduced by Cleary in 2000, where he proved that the group is of type FP_∞. In that paper it is also proved that any element of F_τ can be obtained in this way with subdivisions of the interval.

Binary Trees

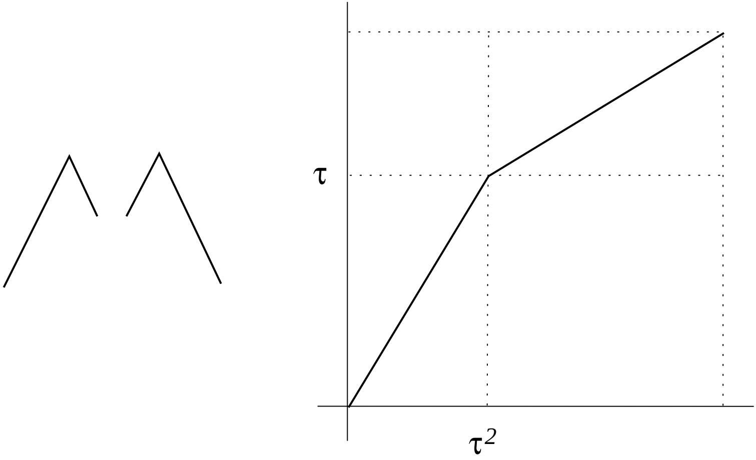
Hence, this opens the possibility to describe elements of F_τ using binary trees, in a similar fashion as it is done for F. A subdivision will be represented by a caret, but we have to distinguish the two types of subdivisions available. This is done using *unbalanced carets.*



The caret on the left is called a y-caret, while the right-hand-side one is an x-caret.

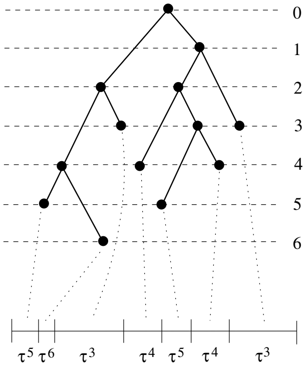
Binary Trees

An example of an element:



Binary Trees

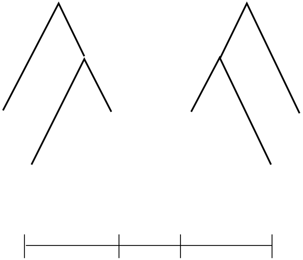
And a tree looks like



where the level of a vertex marks the length of the interval (e.g. a dot at level 3 is an interval of length τ^3).

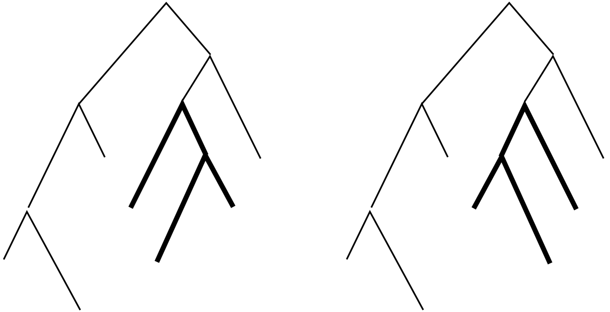
Elements

These subdivisions have a particularity which is important for this group. There exist two trees which represent the same subdivision:



Elements

Hence in a tree we can perform what we call a *basic move.*



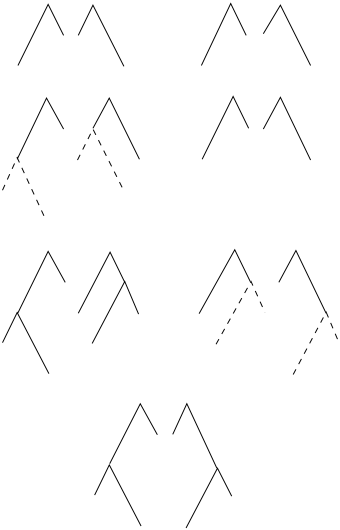
Multiplication

Multiplication is a bit more complicated than in F. Since multiplication is a composition, one needs to match the target tree of the first element with the source tree of the second. But the carets may not match, so the process requires finding subdivisions and using basic moves.

Imagine we need to multiply two copies of the element y_0^{-1}:

Multiplication

Here is the process:



Multiplication

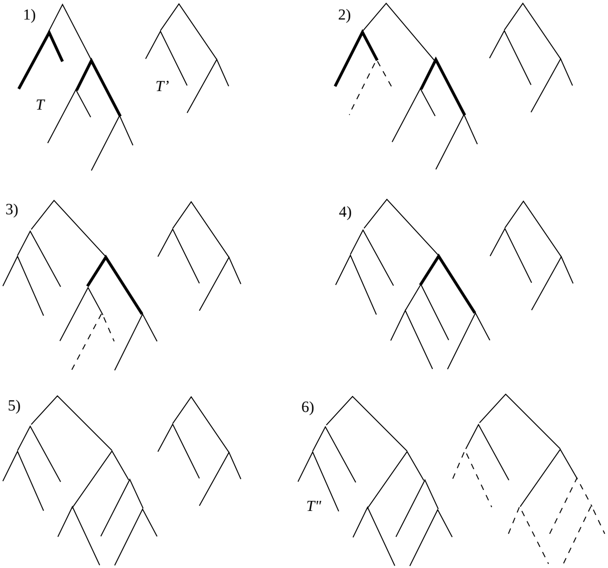
This is guaranteed by the following lemma.

Lemma

Given two trees, we can always find a common subdivision, i.e., a tree which is a subdivision of both.

Obviously this will involve some basic moves. Here is an example of two trees and how to find a common subdivision.

Multiplication



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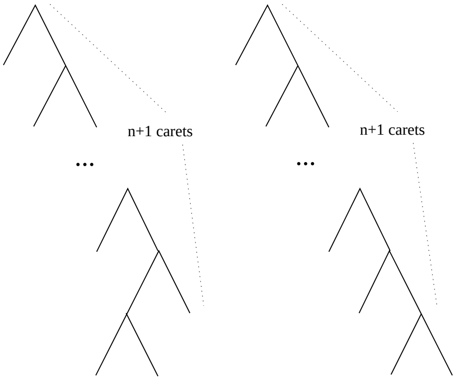
Presentation

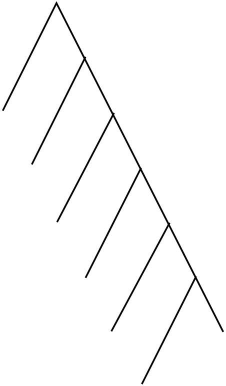
Generators for F

The set of generators for F is the family of elements x_n given by



Generators

We copy these generators for F to obtain those for F_τ. We first define a *spine,* formed by x-carets:



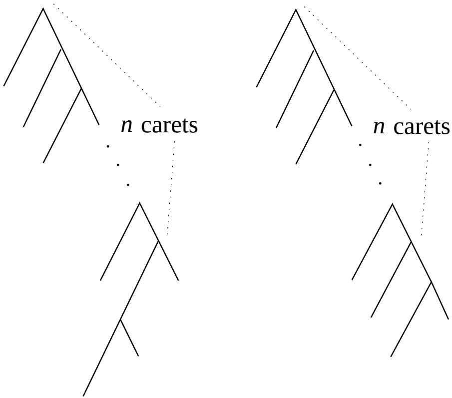
Generators

We observe that the process we used to switch caret types to perform multiplications allows us to change carets on the right hand side of the trees, so that we always have a spine.

Hence instead of mixing types of carets, we only need to use the spine made of x-carets only, and attach an x-caret or a y-caret to obtain a generating set.

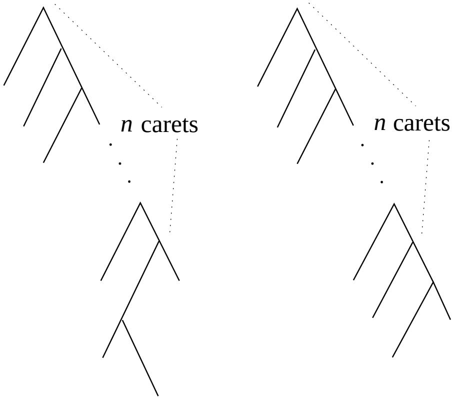
x-generators

This is the generator x_n:



y-generators

And then the second series with a y-caret, creating the generators y_n:



Relations

These generators satisfy the standard sets of relations of the Thompson type, but with both types of generators:

- $x_j x_i = x_i x_{j+1}$
- $x_j y_i = y_i x_{j+1}$
- $y_j x_i = x_i y_{j+1}$
- $y_j y_i = y_i y_{j+1}$

The combinatorics of the trees are the same, this involves only how to attach two carets to a spine. Hence we only need to consider all possibilities for the two carets added.

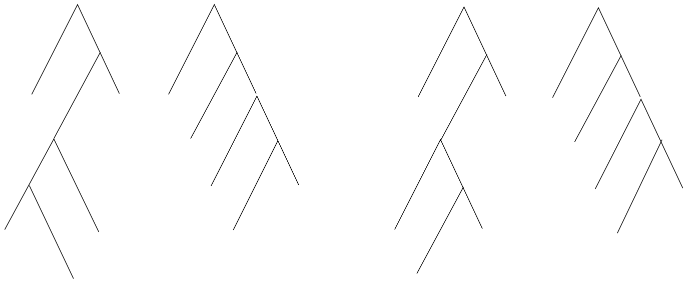
Relations

But they also satisfy a new relation, which is specific for F_τ, representing the basic move, and given by:

$$y_n^2 = x_n x_{n+1}$$

Relations

See, for instance, why the relation $y_1^2 = x_1x_2$ is true. The two elements are the same but because of the basic move they admit two expressions in the generators:



Presentation

Hence we have the presentation:

Theorem (B, Nucinkis, Reeves) A presentation for F_τ is given by the generators x_i, y_i, with the relations

1. $x_j x_i = x_i x_{j+1}$
2. $x_j y_i = y_i x_{j+1}$
3. $y_j x_i = x_i y_{j+1}$
4. $y_j y_i = y_i y_{j+1}$
5. $y_i^2 = x_i x_{i+1}$

for $0 \leq i < j$.

Presentation

To see this is a presentation, we start with a word on the generators which represents the identity. Performing the multiplications we obtain a tree diagram for it, where the two trees represent the same subdivision (since the element is the identity).

We need to show that this element is consequence of the relations. This is shown by the following lemma

Lemma

Given two trees which represent the same subdivision, we can go from one to the other by using only basic moves and without adding any carets.

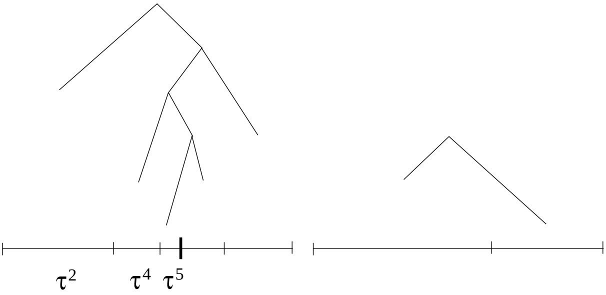
Presentation

Hence each basic move is represented by a relation of type (5), so these relations are sufficient to represent this element. The Thompson relations are necessary to move carets around and perform the basic move relations in the right place of the word.

The proof of this lemma involves using geometric power series based on τ.

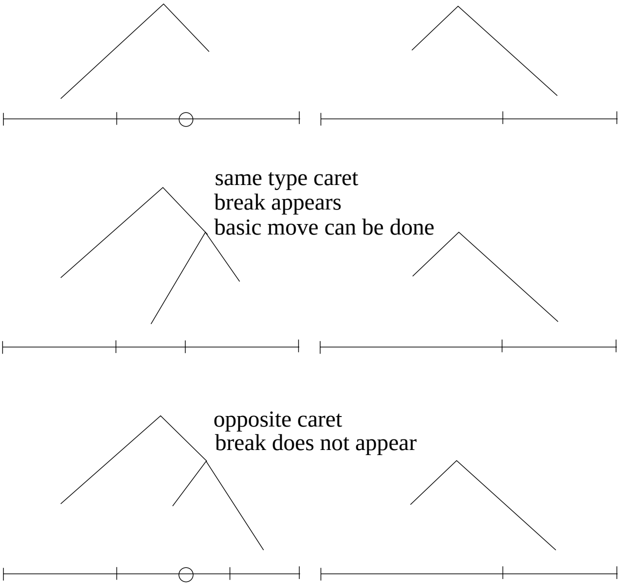
Presentation

Imagine we have two trees representing the same subdivision, but the root carets are of different types.

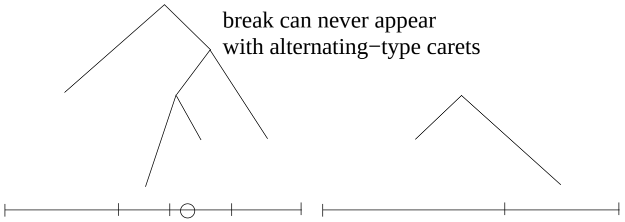


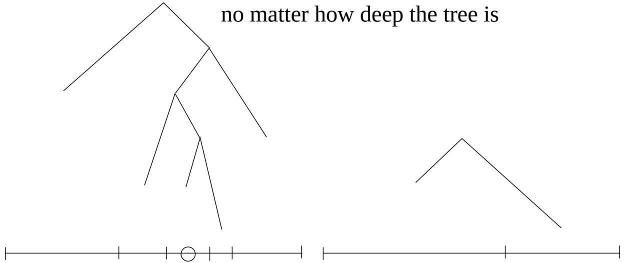
then for the first tree to have a break in the corresponding point given by the second tree, somewhere below it there must be two consecutive carets of the same type so a basic move can be performed.

Presentation



Presentation





Presentation

This process will never end if we do not have the repeated caret type (and hence we can have a basic move). This is because when alternating caret types, we have the following strict inequalities

$$\sum_{k=1}^{n} \tau^{2k} < \tau < 1 - \sum_{k=1}^{n} \tau^{2k+1}$$

and an infinite tree would be required, according to the power series

$$\sum_{k=1}^{\infty} \tau^{2k} = \tau = 1 - \sum_{k=1}^{\infty} \tau^{2k+1}$$

Presentation

Since the trees are finite, if the break must appear in both trees, in one of them there must be two consecutive trees where a basic move can be done.

Hence we only need relations of type (5) and the Thompson relations to transform one tree to the other and then these relations are enough for a presentation.

Normal Form

Construction

The relation $y_i^2 = x_i x_{i+1}$ allows for the construction of an easy normal form.

First of all use the Thompson relators to reorder the generators in the standard fashion. Any element of F_τ admits an expression of the type

$$u_{i_1} u_{i_2} \dots u_{i_n} v_{j_m}^{-1} \dots v_{j_2}^{-1} v_{j_1}^{-1}$$

where

1. the letters u and v represent either x or y,
2. $i_1 \leq i_2 \leq \dots \leq i_n$ and $j_1 \leq j_2 \leq \dots \leq j_m$.

If a higher index appears before a smaller one, use the Thompson relator to switch them. This is the same as in F, but here we get x- and y-generators mixed up.

The special relation

Once all generators with the same index are together, now the special relation comes to our aid. Since we have the relation $y_i^2 = x_i x_{i+1}$, we have that $x_i x_{i+1}$ commutes with y_i:

$$y_i x_i x_{i+1} = y_i^3 = x_i x_{i+1} y_i$$

and this means that

$$y_i x_i = x_i x_{i+1} y_i x_{i+1}^{-1} = x_i y_i x_{i+2} x_{i+1}^{-1}$$

This amounts to $y_i x_i$ being replaced by $x_i y_i$ followed by terms with higher index that are moved farther down the word and are dealt with at a later time.

Only one y

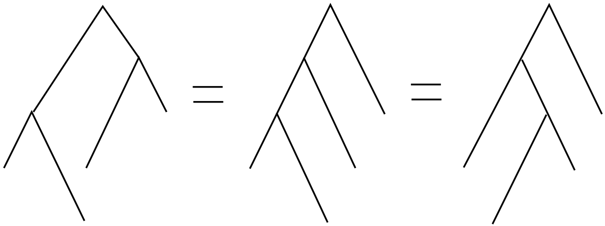
So, in a given subindex, we can accumulate the x-generators first and the y-generators later. Even more, since y_i^2 can be replaced by $x_i x_{i+1}$, we can make it so we have at most only one y-generator per index:

$$x_0^{a_0} y_0^{\epsilon_0} x_1^{a_1} y_1^{\epsilon_1} \dots x_n^{a_n} y_n^{\epsilon_n} \; y_m^{-\delta_m} x_m^{-b_m} \dots y_1^{-\delta_1} x_1^{-b_1} y_0^{-\delta_0} x_0^{-b_0}$$

where $a_i, b_i \geq 0$ and $\epsilon_i, \delta_i \in \{0, 1\}$.

The special relation

This commuting relation between $x_i x_{i+1}$ and y_i has an easy expression in carets:



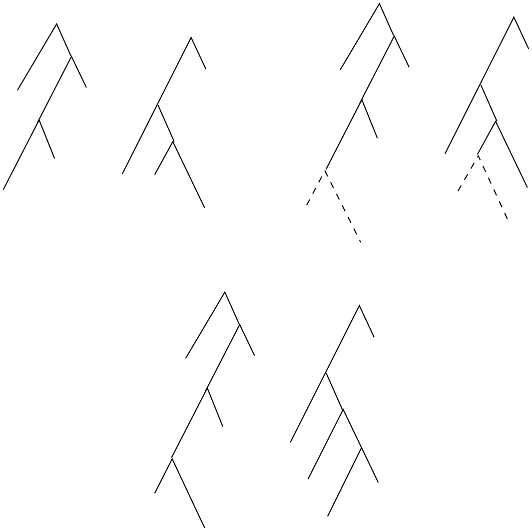
Only ys on the left

The equivalent of the algebraic method above means that all y-carets can be pushed down to the bottom of the tree, i.e. each y-caret has no left children (since these would be x-gens with the same index.

But if a y-caret is at the bottom of the tree with no left children, we can attach another y-caret and convert them to x-generators. Of course y-carets need to be added in both trees. But with this process, we can have the target tree completely free of y-carets.

Only ys on the left

Here is an example:



Seminormal Form

Hence, every element admits an expression in *seminormal form* given by

$$x_0^{a_0} y_0^{\epsilon_0} x_1^{a_1} y_1^{\epsilon_1} \dots x_n^{a_n} y_n^{\epsilon_n} x_m^{-b_m} x_{m-1}^{-b_{m-1}} \dots x_1^{-b_1} x_0^{-b_0}$$

where $a_i, b_i \geq 0$ and $\epsilon_i \in \{0, 1\}$. Observe that the form has only x-generators except for maybe one y-generator per index in the positive part.

Normal Form

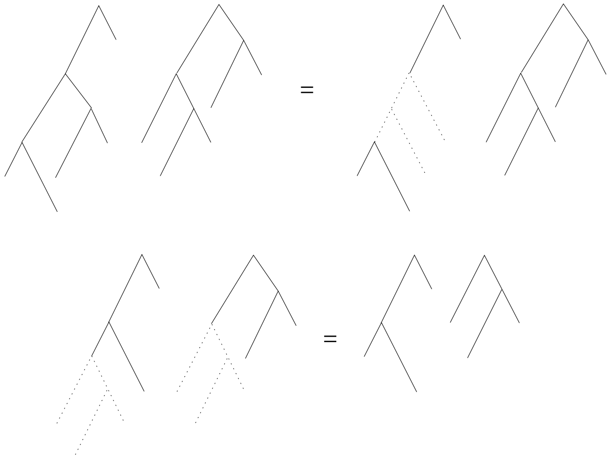
As it happens with F, this seminormal form is not unique. A true normal form needs to satisfy the two extra conditions:

- If a_i and b_i are both nonzero, then at least one of $a_{i+1}, b_{i+1}, \epsilon_i, \epsilon_{i+1}$ is nonzero.
- If w contains a subword of the form $x_i y_i x_{i+2} u x_{i+1}^{-1} x_i^{-1}$, then u contains a generator with index either $i+1$ or $i+2$.

The first condition is standard in Thompson, it corresponds to the tree-pair diagram being reduced. The second condition is specific to F_τ, and corresponds to *hidden cancellations.*

Hidden Cancellations

A hidden cancellation is a cancellation that only appears after a basic move is performed. See an example:



Normal Form

Theorem (B, Nucinkis, Reeves) Every element of F_τ admits a unique normal form.

Commutator and Abelianisation

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Simple Commutator Subgroup

As it happens with F, F_τ satisfies this property.

Theorem (B, Nucinkis, Reeves) The commutator subgroup F_τ' is a simple group.

Simple Commutator Subgroup

The proof of this fact is done the standard way: using a theorem by Higman which constructs simple *second* commutator subgroup due to the high transitivity of the group. Then prove that the first and second commutators are equal, hence showing the commutator is simple.

Abelianisation

Another interesting feature of F_τ is that, due to the special relation $y_i^2 = x_i x_{i+1}$, the abelianisation has torsion.

Theorem (B, Nucinkis, Reeves) The abelianisation of F_τ is isomorphic to $\mathbb{Z}^2 \oplus {}^{\mathbb{Z}}\!/_{2\mathbb{Z}}$.

This is the first known example of group of the Thompson family that has torsion in the abelianisation.

Metric Properties

Metric Estimates

Also similarly to F, the group F_τ admits an estimate of the word metric by carets.

Theorem (B, Nucinkis, Reeves) The word metric can be estimated by the number of carets of the reduced diagram, up to a multiplicative constant.

Distortion

We can consider different subgroups of F_τ which are isomorphic to F. For instance, if we restrict the diagram to have only one type of carets, we obtain the subgroups F_x (whose elements have only x-carets in their diagram) and F_y (with only y-carets). Since the metric on these copies can also be estimated by the number of carets, we immediately obtain the following result:

Theorem (B, Nucinkis, Reeves) The subgroups F_x and F_y are undistorted in F_τ.

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Future research

Future research: T and V

This work opens the doors to several lines of future research. For instance:

The construction can clearly be extended to the T and V versions of F_τ, obtaining groups T_τ and V_τ.

The presentations are similar, just adding the corresponding torsion elements. In particular the relations $y_i^2 = x_i x_{i+1}$ are still valid.

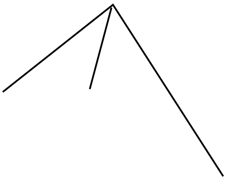
Then these two groups are not simple anymore, but their abelianisation is $\mathbb{Z}/2\mathbb{Z}$ and they both admit index-two subgroups which are now simple.

Future research: other polynomials

Similar groups can be obtained using different polynomials. Jason Brown (L. Reeves's student) has extended these constructions to the groups obtained by considering zeros of the polynomials

$$1 = x^2 + nx$$

where carets have n short legs and one long one. Relations are very similar and everything works in similar fashion.



Future research: other polynomials

For polynomials

$$1 = mx^2 + nx$$

things are more complicated. This is the contents of Nucinkis' student Nick Winstone's thesis. In particular, for these groups not every element is obtained with subdivisions of the intervals. Winstone has a characterisation for those.

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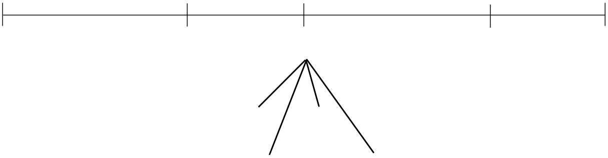
Future research: other polynomials

Even for those where every element is obtained with subdivisions, things are different here.



The reason is that for polynomials such as $1 = 2x^2 + 2x$, the four subintervals are two of each type, and the interval can be split in the middle and this involves rational breakpoints (unlike our case, where they are all irrational). A particularity is that the group of units now has rank two, because both 2 and the irrational root are units.

So this is work in progress.

Thank you!