Continuous-Trace $C^*$-Algebras and the Equivariant Brauer Group

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We let $T$ be a fixed second countable locally compact Hausdorff $G$-space.

We let $\mathcal{S}(T)$ be the collection of all separable $C_0(T)$-algebras, and let $\mathcal{S}_G(T)$ be the collection of pairs $(A, \alpha)$ where $\alpha : G \to \text{Aut} A$ is an action covering the given action on $T$.

We let $S_G(T)$ be the collection of Morita equivalence classes $[A, \alpha]$ over $T$ in $\mathcal{S}_G(T)$.

$S_G(T)$ is a semigroup with respect to the product $[A, \alpha][B, \beta] = [A \otimes C_0(T) B, \alpha \otimes C_0(T) \beta]$ and identity $[C_0(T), \text{Id}]$.

We are going to focus on the subset $\text{Br}_G(T) \subset S_G(T)$ consisting of classes $[A, \alpha]$ where $A$ is a continuous-trace $C^*$-algebra with spectrum $T$.

The classical Dixmier-Douady theory is summarized by observing that $\text{Br}(T) = \text{Br}_{\{e\}}(T)$ is identified with $H^3(T)$ via the map $[A] \mapsto \delta(A)$.
The Equivariant Brauer Group

Theorem (CKRW + aHRW)

The group of invertible elements in $S_G(T)$ is exactly the equivariant Brauer group $\text{Br}_G(T)$. In particular, $\text{Br}_G(T)$ is a group.

Proof.

If $[A, \alpha]$ is invertible in $S_G(T)$, then there is a $B \in \mathcal{S}(T)$ such that $A \otimes_{C_0(T)} B$ is Morita equivalent to $C_0(T)$. A nontrivial result going back to Green implies that in this case both $A$ and $B$ have continuous trace with spectrum $T$. This means that $[A, \alpha] \in \text{Br}_G(T)$. 
Now suppose that $[A, \alpha] \in \text{Br}_G(T)$. We have to exhibit and inverse. As we saw yesterday, the conjugate algebra $\overline{A} = \{ a^b : a \in A \}$ is an inverse for $A$ in $\text{Br}(T)$. Since $A$ is locally Morita equivalent to $C_0(T)$, we can localize and assume there is a $A - C_0(T)$-bimodule $X$ implementing a Morita equivalence between $A$ and $C_0(T)$. We get an element $(\overline{A}, \overline{\alpha}) \in \mathcal{S}_G(T)$ via $\overline{\alpha}_s(a^b) = \alpha_s(a)^b$. We claim that $[\overline{A}, \overline{\alpha}]$ is our desired inverse. This is not so easy. From the classical theory, $X \otimes_{C_0(T)} \overline{X}$ is a $A \otimes_{C_0(T)} \overline{A} - C_0(T)$ bimodule and hence implements a Morita equivalence between $A \otimes_{C_0(T)} \overline{A}$ and $C_0(T)$. 
Proof Continued.

Unfortunately, \( X \otimes_{C_0(T)} \overline{X} \) carries no obvious \( G \)-action—let alone one covering left-translation on \( C_0(T) \). But one can show that \( X \otimes_{C_0(T)} \overline{X} \) is isomorphic (as a Hilbert \( C_0(T) \)-module) to (the completion of)

\[
N = \{ a \in A : t \mapsto \text{tr}(a^*a(t)) \in C_0(T) \}.
\]

(Recall that the fibres of \( A \) are elementary \( C^* \)-algebras and there is a well-defined trace of the positive elements of each fibre.) Unlike \( X \otimes_{C_0(T)} \overline{X} \), \( N \) has a natural \( G \)-action: \( u_s(a) = \alpha_s(a) \). Now it is possible to show that \((N, u)\) implements a Morita equivalence between \((A \otimes_{C_0(T)} \overline{A}, \beta)\) and \((C_0(T), \text{lt})\) for some action \( \beta \). Some serious untangling shows \( \beta = \alpha \otimes_{C_0(T)} \overline{\alpha} \). 

\( \square \)
Remark

Now that we know that $\text{Br}_G(T)$ is a group, we can set about trying to pin down its properties. As we shall show, this group has a rich structure that encapsulates an number of interesting properties of group actions on continuous-trace $C^*$-algebras. Let’s start by looking at some special cases.
If $G = \{ e \}$ is trivial, then $\text{Br}_G(T)$ is the ordinary Brauer group we studied in the first part of yesterday’s lecture.

What we saw was that $\text{Br}(T)$—which now consists of Morita equivalence classes (over $T$) of continuous-trace $C^*$-algebras with spectrum $T$—summarizes the Dixmier-Douady theory for continuous trace $C^*$-algebras with spectrum $T$.

In this form, the results are due to Green.

**Theorem (Last Time; DD & Green)**

The map sending $[A] \in \text{Br}(T)$ to its Dixmier-Douady class $\delta(A) \in H^3(T)$ is a group isomorphism. In particular, $\delta(A \otimes_{C_0(T)} B) = \delta(A) + \delta(B)$ and $\delta(\overline{A}) = -\delta(A)$ where $\overline{A}$ is the conjugate algebra.
Another Extreme case: \( T = \{ \ast \} \)

- Now we take the case where the space \( T \) is reduced to a point.
- Then we are effectively considering homomorphisms \( \alpha : G \to \text{Aut } K \) which are determined by projective representations \( u : G \to P(U(\mathcal{H})) = U(\mathcal{H})/T \).
- Projective representations are determined by cocycle representations \( \pi : G \to U(\mathcal{H}) \) where \( \omega(s, t)\pi(st) = \pi(s)\pi(t) \) with \( \omega(s, t) \in T \).
- The class \( c(\alpha) \) of \( \omega \) in \( H^2(G, T) \) is called the Mackey obstruction for \( \alpha \) or \( u \) and vanishes exactly when we can lift \( u \) to a homomorphism into \( U(\mathcal{H}) \).
- We can think of \( H^2(G, T) \) as the group of isomorphism classed of central group extensions \( 1 \to T \to E \to G \to \{ e \} \).

**Theorem**

*Classes in \( \text{Br}_G(\ast) \) are given by pairs \([K, \alpha]\) and the Mackey obstruction map \([K, \alpha] \mapsto c(\alpha)\) is a group isomorphism of \( \text{Br}_G(\ast) \) onto \( H^2(G, T) \).*
Foreshadowing

Remark
Later I will argue that the general equivariant Brauer group $\text{Br}_G(T)$ is “blend” of the previous two extreme cases: first where the group is trivial and second where the space is trivial. But first, more special cases.
An action of $G$ on $T$ is called proper if $(s, x) \mapsto (s \cdot x, x)$ is proper in that the inverse image of compact sets is compact.

If $G$ acts properly on $T$, then the orbit space $G \backslash T$ is locally compact and Hausdorff.

Our next special case is an old result of Raeburn and Rosenberg reformulated in our context.

**Theorem (RR 88)**

Suppose that $G$ acts freely and properly on $T$. Then the map $[A, \alpha] \mapsto [A \rtimes_{\alpha} G]$ is an isomorphism of $\text{Br}_G(T)$ onto $\text{Br}(G \backslash T)$. If $[B] \in \text{Br}(G \backslash T)$ then the inverse sends $[B]$ to the class of the pull-back $(C_0(T) \otimes_{C_0(G \backslash T)} B, \text{lt} \otimes 1)$ by the orbit map $p : T \to G \backslash T$. Therefore the map $[A, \alpha] \mapsto \delta(A \rtimes_{\alpha} G)$ is a group isomorphism of $\text{Br}_G(T)$ onto $H^3(G \backslash T)$. 
It is worth a moment to see what is being claimed here for a free and proper action of $G$ on $T$.

First, $[A, \alpha] \in \text{Br}_G(T)$ implies that $A \rtimes_\alpha G$ has continuous trace with spectrum $G \setminus T$.

Also $A$ is Morita equivalent to $C_0(T) \otimes_{C_0(G \setminus T)} (A \rtimes_\alpha G)$ and $\delta(A) \in p^*(H^3(G \setminus T))$ for the orbit map $p : T \to G \setminus T$.

Given any class $c \in H^3(G \setminus T)$, there is a continuous-trace $C^*$-algebra $A$ with $\delta(A) = p^*(c)$ admitting a $G$-action covering the given free and proper action on $T$.

It follows that stable continuous-trace $C^*$-algebra $A$ (that is, $A \otimes \mathcal{K} \cong A$) admits an action covering the given free and proper action on $T$ if and only if $\delta(A) \in p^*(H^3(G \setminus T))$. 
An remarkably fertile class of examples is that case where the $G$ action on $T$ is trivial: $s \cdot t = t$ for all $s \in G$ and $t \in T$.

It was Judy Packer that first made progress in this direction. Subsequent players include Siegfried Echterhoff, Ryszard Nest, Iain Raeburn, and myself.

The first observation is that there is always a natural homomorphism $F : \text{Br}_G(T) \to \text{Br}(T)$ sending $[A, \alpha] \mapsto [A]$—this called the forgetful homomorphism—and it will play an important role here and down the road.

In the case of trivial actions, $F : \text{Br}_G(T) \to \text{Br}(T)$ has a natural splitting and hence $\text{Br}_G(T) \cong \text{Br}(T) \oplus \ker F$.

It is not hard to identify $\ker F$ with the group $\mathcal{E}_G(T)$ of exterior equivalence class of $C_0(T)$-actions on $C_0(T, \mathcal{K})$. 

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If \([C_0(T, \mathcal{K}), \alpha]\) is a \(C_0(T)\)-action in \(\mathcal{E}_G(T)\), then evaluation at \(t \in T\) gives an action \(\alpha^t\) on \(\mathcal{K}\).

This gives us a well defined map \(\varphi^\alpha : T \to H^2(G, T)\) where \(\varphi^\alpha(t)\) is the Mackey obstruction \(c(\alpha^t)\) for \(\alpha^t : T \to \text{Aut}\mathcal{K}\).

It can be shown that \(\varphi^\alpha\) is continuous and we get a map \(\Phi : \mathcal{E}_G(T) \to C(T, H^2(G, T))\).

In general, \(\ker \Phi\) consists of (exterior equivalence classes) of pointwise unitary automorphism groups and does not have an elegant description.

However, under certain conditions, for example if \(H^2(G, T)\) is Hausdorff and the abelianization \(G_{ab} = G/[G, G]\) is compactly generated, Rosenberg has shown that \(\ker \Phi\) consists of locally unitary automorphisms.

Then work of Phillips & Raeburn shows that \(\ker \Phi\) is parameterized by isomorphism classes of principal \(\hat{G}_{ab}\)-bundles over \(T\) which are in turn parameterized by the sheaf cohomology group \(H^1(T, \hat{G}_{ab})\).
A good description of $\mathcal{E}_G(T)$ requires both a description of the kernel of $\Phi : \mathcal{E}_G(T) \to C(T, H^2(G, T))$ and a splitting map. Packer and later Packer & Raeburn showed $\Phi$ splits if $G$ is an elementary abelian group ($G = \mathbb{R}^n \times T^m \times \mathbb{Z}^k \times F$ where $F$ is finite). Later Echterhoff and I extended this to “smooth” groups which include connected, simply connected Lie groups; connected semi-simple Lie groups; as well as all compact groups; discrete groups; and compactly generated abelian groups.
The Result

**Theorem (EW)**

Suppose that $G$ is smooth as on the previous slide. Suppose that $G$ acts trivially on $T$ and that $G_{ab}$ is compactly generated. Then

$$\text{Br}_G(T) \cong \text{Br}(T) \oplus C(T, H^2(G, T)) \oplus H^1(T, \hat{G}_{ab}).$$

**Proof.**

Because the action is trivial, the forgetful homomorphism is surjective and we always have $\text{Br}_G(T) = \text{Br}(T) \oplus \ker(F)$. We can identify $\ker F$ with $\mathcal{E}_G(T)$. If $G$ is smooth, the map $\Phi : \mathcal{E}_G(T) \to C(T, H^2(G, T))$ is surjective and splits. Then we can identify $\ker \Phi$ as on the previous slide.
Let’s Take a Short Break
The Range of the Forgetful Homomorphism

Remark

In the general case, the forgetful homomorphism $F : \text{Br}_G(T) \to \text{Br}(T) \cong H^3(T)$ will not be surjective. Nevertheless, the range is interesting not only to get a picture of the structure of $\text{Br}_G(T)$, but because it allows us to determine which stable continuous-trace $C^*$-algebras with spectrum $T$ admit actions covering the given action on $T$. As I will show, our analysis yields a very interesting and nontrivial answer to the later question.
Remark (Step One)

Note that $H^n(T)$ is a $G$-module. By functorality, $\text{lt}_s \in \text{Homeo}(T)$ induces an isomorphism $(\text{lt}_s)^*$ and we let $s \cdot c = (\text{lt}_s)^*(c)$ for $c \in H^n(T)$. Thus if $\alpha \in \text{Aut} A$ and $A$ is a continuous-trace $C^*$-algebra with spectrum $T$, then $\alpha$ induces a homomorphism $\alpha_*$ of $T$, and hence an isomorphism $(\alpha_*)^*$ of $H^3(T)$. An old result of Phillips & Raeburn asserts that $(\alpha_*)^*(\delta(A)) = \delta(A)$. This observation gives us our first obstruction for a class $c \in H^3(T)$ to lie in the range, $\text{Im} F$, of the forgetful homomorphism.

Proposition (First Obstruction)

If $c \in \text{Im}(F)$, then

$$c \in H^3(T)^G = \{ c \in H^3(T) : s \cdot c = c \text{ for all } s \in G \}.$$
To further pin down $\text{Im}(F)$, we need some more basic work of Phillips & Raeburn. Let $[A] \in \text{Br}(T)$ and let $\text{Aut}_{C_0(T)} A$ be the $C_0(T)$-linear automorphisms of $A$—which are necessarily locally inner. Then there is an obstruction $\zeta(\alpha)$ in $H^1(T, S) \cong H^2(T)$ to $\alpha$ being inner. Then if $A$ is stable, we have a short exact sequence

$$1 \longrightarrow \text{Inn} A \longrightarrow \text{Aut}_{C_0(T)} A \longrightarrow H^2(T) \longrightarrow 0$$

Let $\text{Out} A = \text{Aut}(A)/\text{Inn} A$. Then we also get a short exact sequence

$$1 \longrightarrow \text{Aut}_{C_0(T)} A/\text{Inn} A \longrightarrow \text{Out} A \longrightarrow \text{Homeo}_{\delta(A)}(T) \longrightarrow 1$$

where $\text{Homeo}_{\delta(A)}(T)$ are the homeomorphisms $h$ of $T$ such that $h^*(\delta(A)) = \delta(A)$.  

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Using the short exact sequences on the previous side, we can identify $\text{Aut}_{C_0(T)} A/\text{Inn} A$ with $H^2(T)$ and $\text{Out} A/\text{Im}(H^2(T))$ with $\text{Homeo}_{\delta(A)}$ (see ses). If $c \in H^3(T)^G$, then there is an essentially unique stable $A$ such that $\delta(A) = c$. But assumption on $c$, the given $G$-action on $T$ gives us a homomorphism $\lambda : G \to \text{Homeo}_{\delta(A)}(T)$. Thus we get a diagram

\[
1 \longrightarrow H^2(T) \longrightarrow \text{Out} A \longrightarrow \text{Out} A/\text{Im}(H^2(T)) \longrightarrow 1
\]

If we can find an action covering $\lambda$, then we can find $\lambda$ such that the diagram commutes. General nonsense there is an obstruction to doing this in the group cohomology group $H^2(G, H^2(T))$.

Making this work for the Moore groups requires some hypotheses.
Lemma (Second Obstruction)

Suppose that $H^2(T)$ is countable. (For example, this is automatic if $T$ has the homotopy type of a compact metric space.) Then there is a homomorphism $d_2 : H^3(T)^G \rightarrow H^2(G, H^2(T))$ such that $d_2(c) = 0$ if and only if the map $\text{lt} : G \rightarrow \text{Homeo}_c(T)$ lifts to a continuous map $\lambda : G \rightarrow \text{Out}(A)$ on the previous slide.
We now suppose that $c \in H^3(T)^G$ and $d_2(c) = 0$. Then we want to lift $\lambda$ so that

\[
\begin{array}{ccc}
\text{Aut } A & \xrightarrow{\alpha} & \text{Out } A \\
\downarrow q & & \downarrow \lambda \\
G & \xrightarrow{\lambda} & \text{Out } A
\end{array}
\]

commutes for $A$ with $\delta(A) = c$. Describing this obstruction a bit technical, so I will only crudely describe it here. Using $\lambda$, $C(X, T)$ becomes a $G$-module and we can form the Moore group $H^3(G, C(T, T))$. Using Busby-Smith twisted actions and the “Packer-Raeburn Stabilization Trick”, we can find an element $d_A(\lambda) \in H^3(G, C(T, T))$ which vanishes when we can lift $\lambda$ as above. Unfortunately, we have to worry about the choices of $A$ and $\lambda$. It turns out that there is a homomorphism $d_2' : H^1(G, H^2(T)) \to H^3(G, C(T, T))$ so that we obtain as well-defined homomorphism $d_3 : \ker d_2 \to H^3(G, C(T, T))/\text{Im } d_2'$ so that if $c \in \text{Im } F$, $d_3(c) = 0$. 

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Theorem (CKRW)

If \( H^2(T) \) is countable, then the image of the forgetful homomorphism \( F : \text{Br}_G(T) \to H^3(T) \) consists of classes \( c \in H^3(T)^G \) such that \( d_2(c) = 0 = d_3(c) \). In particular, a given stable continuous-trace \( C^* \)-algebra \( A \) with spectrum \( T \) admits a \( G \)-action covering the given action on \( T \) if and only if \( \delta(A) \in H^3(T)^G \) and \( d_2(\delta(A)) = d_3(\delta(A)) = 0 \).
Corollary

Suppose that $T$ is a $\mathbb{R}$-space with $H^0(T)$, $H^1(T)$, and $H^2(T)$ all countable. If $A$ is a stable continuous-trace $C^*$-algebra with spectrum $T$, then there is, up to exterior equivalence, exactly one action of $\alpha : \mathbb{R} \to \text{Aut} A$ covering the given action on $T$.

Proof.

Since $\mathbb{R}$ is connected, $\text{lt}_0$ and $\text{lt}_s$ are homotopic for all $s \in \mathbb{R}$ and $H^3(T) = H^3(T)^{\mathbb{R}}$. Since $H^2(T)$ is countable, Wigner has shown $H^2(\mathbb{R}, H^2(T))$ is trivial. Raeburn and Rosenberg have shown that $H^3(\mathbb{R}, C(T, T))$ is trivial under our assumptions on $H^* (T)$. Hence all our obstructions in the previous theorem vanish and $F$ is surjective. However $H^3(\mathbb{R}, C(T, T))$ trivial also implies that there can be at most one (up to exterior equivalence) action. Hence the result.
Remark

The corollary on the previous slide implies that ker $F$ is trivial. In general, since $F$ is a group homomorphism, ker $F$ parameterizes (up to exterior equivalence) how many actions cover a given action on the spectrum of a continuous-trace $C^*$-algebra whenever that Dixmier-Douady class is in the range of $F$. Our next goal is to see what we can say in general about ker $F$ and the structure of $\text{Br}_G(T)$. It should be acknowledged that the motivation for all this—including our description of Im $F$—comes from work of Kumjian and Parker. Kumjian in particular, used a spectral sequence of Grothendieck’s which predicts many of the constructions here and below—even though no known generalization of Grothendieck’s spectral sequence is known for non-discrete groups.
We are able to give a filtration \( \{ e \} \subset B_1 \subset B_2 \subset Br_G(T) \) and then describe the quotients \( Br_G(G)/B_2 \) and \( B_2/B_1 \) in terms of Moore group cohomology groups. We will still need \( H^2(T) \) countable throughout. A full discussion would take us too far afield, so we will settle for a crude summary here.

We let \( B_2 = \ker F \). Thus \( B_2 \) is the subgroup \( \mathcal{O}_G(T) \) of classes in \( Br_G(T) \) of the form \([C_0(T, K), \alpha]\). We have already completely described

\[
Br_G(T)/ \ker F = \text{Im } F
\]

as the subgroup of \( H^3(T)^G \) determined by the homomorphisms \( d_2 \) and \( d_3 \).
The Homomorphism $\eta$

If $[C_0(T, K), \alpha] \in \ker F$, then $\alpha$ induces the given action of $G$ on $T$. Then for each $s \in G$, $\alpha_s \circ (\text{lt}_s^{-1} \otimes 1) \in \text{Aut}_{C_0(T)} C_0(T, K)$. The Phillips–Raeburn obstruction to this automorphism being inner is in $H^2(T)$ and gives us a 1-cocycle from $G$ to $H^2(T)$. Hence we get a homomorphism $\eta : \ker F \to H^1(G, H^2(T))$ whose range coincides with $\ker d'_2 : H^1(G, H^2(T)) \to H^3(G, C(T, T))$ we encountered earlier. Thus $B_2/B_1 = \ker F/\ker \eta$ is identified with $\ker d'_2 \subset H^1(G, H^2(T))$. We can also show that $B_1 = \ker \eta$ is isomorphic to a quotient of $H^2(G, C(T, T))$.

That’s all I want to say here about $\text{Br}_G(T)$. 

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What About $S_G(T)$?

To get some information about $S_G(T)$ requires restriction to nice actions of $G$ on $T$. First, we should recall what Rieffel-proper actions are. For simplicity we assume $G$ is unimodular.

**Definition**

A dynamical system $(A, G, \alpha)$ is called **Rieffel-proper** if there is a dense $\alpha$-invariant subalgebra $A_0$ of $A$ such that

1. for all $a, b \in A_0$, $s \mapsto \langle a, b \rangle(s) := a\alpha_s(b^*)$ is in $L^1(G, A)$,

2. For each $a, b \in A_0$ there is a unique $\alpha$-invariant multiplier $\langle a, b \rangle_\ast \in M(A)^\alpha$ such that for all $c \in A_0$ we have

$$\int_G c\alpha_s(a^*b)\,ds = c\langle a, b \rangle_\ast.$$

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We let $E_0 = \text{span}\{ \langle a, b \rangle : a, b \in A_0 \}$. Then $E_0$ is an ideal in $A \rtimes_{\alpha, r} G$. We call the action saturated if $E_0$ is dense in $A \rtimes_{\alpha, r} G$. The closure $A^\alpha$ in $M(A)$ of $\text{span}\{ \langle a, b \rangle : a, b \in A_0 \}$ is called the generalized fixed point algebra of $\alpha$.

**Theorem (Rieffel, Buss-Echterhoff)**

Suppose that $(A, G, \alpha)$ is a saturated Rieffel-proper dynamical system. Then we can complete $A_0$ to a bimodule implementing a Morita equivalence between $A \rtimes_{\alpha, r} G$ and the generalized fixed point algebra $A^\alpha$. 
If $G$ acts freely and properly on $T$ and if $(A, \alpha) \in \mathcal{S}_G(T)$, then $\alpha$ is Rieffel-proper and saturated.

Moreover, in this case $A \rtimes_\alpha G = A \rtimes_{\alpha, r} G$, and $A \rtimes_\alpha G$ is Morita equivalent to $A^\alpha$ over $G \setminus T$.

Thus we obtain a semigroup homomorphism $[A, \alpha] \mapsto [A^\alpha]$ from $S_G(T)$ to $S(G \setminus T)$.

Even better, the pull-back $B \mapsto (C_0(T) \otimes_{C_0(G \setminus T)} B, \text{lt} \otimes \text{id})$ turns out to implement an inverse.

Hence the following result due to an Huef, Raeburn, and myself.
Proposition (aHRW)

Suppose that $G$ acts freely and properly on $T$. If $(A, \alpha) \in \mathcal{S}_G(T)$, then $\alpha$ is Rieffel-proper and saturated. Furthermore, $A^\alpha$ is Morita equivalent to $A \rtimes^\alpha G$ and the map $[A, \alpha] \mapsto [A^\alpha]$ is a semigroup isomorphism of $S_G(T)$ onto $S(G \setminus T)$.

Remark (Special Cases)

1. If $A = C_0(T)$, then the properness implies $G \setminus T$ is Hausdorff and then $A^\alpha = C_0(G \setminus T)$. Furthermore $C_0(T) \rtimes_{lt} G$ is Morita equivalent to $C_0(G \setminus T)$.

2. If $A = C_0(T, D)$ and $(D, G, \alpha)$ is any dynamical system, then $(A, \gamma) = (C_0(T, D), \text{lt} \otimes \alpha) \in \mathcal{S}_G(T)$ and $A^\gamma = \text{Ind}_G^T(D, \alpha)$ where the later is the induced algebra consisting of continuous functions $f : T \to D$ such that $f(s^{-1} \cdot t) = \alpha_s(f(t))$ and $G \cdot t \mapsto \|f(t)\|$ vanishes at infinity.
The proposition on the previous slide has an equivariant version.

Suppose we have commuting free and proper actions of $G$ (on the left) and $H$ (on the right) of $T$.

Suppose $(A, \alpha \times \beta) \in S_{G \times H}(T)$.

Then $\beta$ extends to $M(A)$ and restricts to $\bar{\beta}$ on $A^\alpha$, and $\bar{\beta}$ turns out to be continuous on $A^\alpha$.

Then $[A, \alpha \times \beta] \mapsto [A^\alpha, \bar{\beta}]$ gives an isomorphism of $S_{G \times H}(T)$ onto $S_H(G \setminus T)$.

The situation is symmetric in $G$ and $H$ so we can prove the following.
The Theorem

**Theorem (aHRW)**

Suppose that we have commuting free and proper actions of $G$ and $H$ on $T$ as above. Then there is a semigroup isomorphism

$$\theta : S_G(T/H) \rightarrow S_H(G\setminus T)$$

such that if $[B, \beta] = \theta([A, \alpha])$, then $A \rtimes_\alpha G$ is Morita equivalent to $B \rtimes_\beta H$. Indeed, every class in $S_G(T/H)$ is of the form $[A^\beta, \bar{\alpha}]$ for some $(A, \alpha \times \beta) \in \mathcal{G} \times H(T)$, and we have

$$\theta([A^\beta, \bar{\alpha}]) = [A^\alpha, \bar{\beta}],$$

so that $A^\beta \rtimes_{\bar{\alpha}} G$ is Morita equivalent to $A^\alpha \rtimes_{\bar{\beta}} H$. 
Taking $A = C_0(T)$ gives the Morita equivalence between $C_0(G\setminus T) \rtimes H$ and $C_0(T/H) \ltimes G$ usually called Green’s Imprimitivity Theorem.

Since a semigroup isomorphism preserves the subgroups of invertible elements, we also obtain the following result due to Kumjian, Raeburn, and myself.

**Theorem**

Suppose $G$, $H$, and $T$ are as above. Then $\theta$ restricts to a group isomorphism of $\text{Br}_G(T/H)$ onto $\text{Br}_H(G\setminus T)$. 
THANK YOU
References


Siegfried Echterhoff and Dana P. Williams, *Locally inner actions on $C_0(X)$-algebras*, J. Operator Theory **45** (2001), no. 1, 131–160. MR 1 823 065


