Continuous-Trace C^*-Algebras and the Equivariant Brauer Group

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Restart

- We let T be a fixed second countable locally compact Hausdorff G-space.
- We let $\mathscr{S}(T)$ be the collection of all separable $C_0(T)$-algebras, and let $\mathscr{S}_G(T)$ be the collection of pairs (A, α) where $\alpha : G \to \operatorname{Aut} A$ is an action covering the given action on T.
- We let $S_G(T)$ be the collection of Morita equivalence classes $[A, \alpha]$ over T in $\mathscr{S}_G(T)$.
- $S_G(T)$ is a semigroup with respect to the product $[A, \alpha][B, \beta] = [A \otimes_{C_0(T)} B, \alpha \otimes_{C_0(T)} \beta]$ and identity $[C_0(T), \mathrm{lt}]$.
- We are going to focus on the subset $\mathrm{Br}_G(T) \subset S_G(T)$ consisting of classes $[A, \alpha]$ where A is a continuous-trace C^*-algebra with spectrum T.
- The classical Dixmier-Douady theory is summarized by observing that $\mathrm{Br}(T) = \mathrm{Br}_{\{e\}}(T)$ is identified with $H^3(T)$ via the map $[A] \mapsto \delta(A)$.

Theorem (CKRW + aHRW)

The group of invertible elements in $S_G(T)$ is exactly the equivariant Brauer group $\mathrm{Br}_G(T)$. In particular, $\mathrm{Br}_G(T)$ is a group.

Proof.

If $[A, \alpha]$ is invertible in $S_G(T)$, then there is a $B \in \mathscr{S}(T)$ such that $A \otimes_{C_0(T)} B$ is Morita equivalent to $C_0(T)$. A nontrivial result going back to Green implies that in this case both A and B have continuous trace with spectrum T. This means that $[A, \alpha] \in \mathrm{Br}_G(T)$.

The Converse

Proof Continued.

Now suppose that $[A, \alpha] \in \mathrm{Br}_G(T)$. We have to exhibit and inverse. As we saw yesterday, the conjugate algebra $\overline{A} = \{\, a^\flat : a \in A \,\}$ is an inverse for A in $\mathrm{Br}(T)$. Since A is locally Morita equivalent to $C_0(T)$, we can localize and assume there is a $A\!-C_0(T)$-bimodule X implementing a Morita equivalence between A and $C_0(T)$. We get an element $(\overline{A}, \overline{\alpha}) \in \mathscr{S}_G(T)$ via $\overline{\alpha}_s(a^\flat) = \alpha_s(a)^\flat$. We claim that $[\overline{A}, \overline{\alpha}]$ is our desired inverse. This is not so easy. From the classical theory, $\mathsf{X} \otimes_{C_0(T)} \overline{\mathsf{X}}$ is a $A \otimes_{C_0(T)} \overline{A}\!- C_0(T)$ bimodule and hence implements a Morita equivalence between $A \otimes_{C_0(T)} \overline{A}$ and $C_0(T)$.

Proof

Proof Continued.

Unfortunately, $\mathsf{X} \otimes_{C_0(T)} \overline{\mathsf{X}}$ carries no obvious G-action—let alone one covering left-translation on $C_0(T)$. But one can show that $\mathsf{X} \otimes_{C_0(T)} \overline{\mathsf{X}}$ is isomorphic (as a Hilbert $C_0(T)$-module) to (the completion of)

$$\mathsf{N} = \{\, a \in A : t \mapsto \operatorname{tr}(a^*a(t)) \in C_0(T)\,\}.$$

(Recall that the fibres of A are elementary C^*-algebras and there is a well-defined trace of the positive elements of each fibre.) Unlike $\mathsf{X} \otimes_{C_0(T)} \overline{\mathsf{X}}$, N has a natural G-action: $u_s(a) = \alpha_s(a)$. Now it is possible to show that (N, u) implements a Morita equivalence between $(A \otimes_{C_0(T)} \overline{A}, \beta)$ and $(C_0(T), \mathrm{lt})$ for some action β. Some serious untangling shows $\beta = \alpha \otimes_{C_0(T)} \overline{\alpha}$. □

Special Cases

Remark

Now that know that $\mathrm{Br}_G(T)$ is a group, we can set about trying to pin down its properties. As we shall show, this group has a rich structure that encapsulates an number of interesting properties of group actions on continuous-trace C^*-algebras. Let's start by looking at some special cases.

G Trivial

- If $G = \{ e \}$ is trivial, then $\mathrm{Br}_G(T)$ is the ordinary Brauer group we studied in the first part of yesterday's lecture.
- What we saw was that $\mathrm{Br}(T)$—which now consists of Morita equivalence classs (over T) of continuous-trace C^*-algbras with spectrum T—summarizes the Dixmier-Douady theory for continuous trace C^*-algebras with spectrum T.
- In this form, the results are due to Green.

Theorem (Last Time; DD & Green)

The map sending $[A] \in \mathrm{Br}(T)$ *to it's Dixmier-Douady class* $\delta(A) \in H^3(T)$ *is a group isomorphism. In particular,* $\delta(A \otimes_{C_0(T)} B) = \delta(A) + \delta(B)$ *and* $\delta(\overline{A}) = -\delta(A)$ *where* $\overline{A}$ *is the conjugate algebra.*

Another Extreme case: $T = \{ * \}$

- Now we take the case where the space T is reduced to a point.
- Then we are effectively considering homomorphisms $\alpha : G \to \operatorname{Aut} K$ which are determined by projective representations $u : G \to P(U(\mathcal{H})) = U(\mathcal{H})/\mathbf{T}$.
- Projective representations are determined by cocycle representations $\pi : G \to U(\mathcal{H})$ where $\omega(s,t)\pi(st) = \pi(s)\pi(t)$ with $\omega(s,t) \in \mathbf{T}$.
- The class $c(\alpha)$ of ω in $H^2(G, \mathbf{T})$ is called the Mackey obstruction for α or u and vanishes exactly when we can lift u to a homomorphism into $U(\mathcal{H})$.
- We can think of $H^2(G, \mathbf{T})$ as the group of isomorphism classed of central group extensions $1 \to T \to E \to G \to \{ e \}$.

Theorem

Classes in $\mathrm{Br}_G(*)$ are given by pairs $[\mathcal{K}, \alpha]$ and the Mackey obstruction map $[\mathcal{K}, \alpha] \mapsto c(\alpha)$ is a group isomorphism of $\mathrm{Br}_G(*)$ onto $H^2(G, \mathbf{T})$.

Foreshadowing

Remark

Later I will argue that the general equivariant Braurer group $\mathrm{Br}_G(T)$ is "blend" of the previous two extreme cases: first where the group is trivial and second where the space is trivial. But first, more special cases.

Free and Proper Actions

- An action of G on T is called proper if $(s, x) \mapsto (s \cdot x, x)$ is proper in that the inverse image of compact sets is compact.
- If G acts properly on T, then the orbit space $G \backslash T$ is locally compact and Hausdorff.
- Our next special case is an old result of Raeburn and Rosenberg reformulated in our context.

Theorem (RR 88)

Suppose that G acts freely and properly on T. Then the map $[A, \alpha] \mapsto [A \rtimes_\alpha G]$ is an isomorphism of $\mathrm{Br}_G(T)$ onto $\mathrm{Br}(G \backslash T)$. If $[B] \in \mathrm{Br}(G \backslash T)$ then the inverse sends $[B]$ to the class of the pull-back $(C_0(T) \otimes_{C_0(G\backslash T)} B, \mathrm{lt} \otimes 1)$ by the orbit map $p : T \to G \backslash T$. Therefore the map $[A, \alpha] \mapsto \delta(A \rtimes_\alpha G)$ is a group isomorphism of $\mathrm{Br}_G(T)$ onto $H^3(G \backslash T)$.

Showing Off

- It is worth a moment to see what is being claimed here for a free and proper action of G on T.
- First, $[A, \alpha] \in \mathrm{Br}_G(T)$ implies that $A \rtimes_\alpha G$ has continuous trace with spectrum $G\backslash T$.
- Also A is Morita equivalent to $C_0(T) \otimes_{C_0(G\backslash T)} \bigl(A \rtimes_\alpha G\bigr)$ and $\delta(A) \in p^*\bigl(H^3(G\backslash T)\bigr)$ for the orbit map $p : T \to G\backslash T$.
- Given any class $\mathfrak{c} \in H^3(G\backslash T)$, there is a continuous-trace C^*-algebra A with $\delta(A) = p^*(\mathfrak{c})$ admitting a G-action covering the given free and proper action on T.
- It follows that stable continuous-trace C^*-algebra A (that is, $A \otimes \mathcal{K} \cong A$) admits an action covering the given free and proper action on T if and only if $\delta(A) \in p^*\bigl(H^3(G\backslash T)\bigr)$.

Trivial Actions

- An remarkably fertile class of examples is that case where the G action on T is trivial: $s \cdot t = t$ for all $s \in G$ and $t \in T$.
- It was Judy Packer that first made progress in this direction. Subsequent players include Siegfried Echterhoff, Ryszard Nest, Iain Raeburn, and myself.
- The first observation is that there is always a natural homomorphism $F : \mathrm{Br}_G(T) \to \mathrm{Br}(T)$ sending $[A, \alpha] \mapsto [A]$—this called the forgetful homomorphism—and it will play an important role here and down the road.
- In the case of trivial actions, $F : \mathrm{Br}_G(T) \to \mathrm{Br}(T)$ has a natural splitting and hence $\mathrm{Br}_G(T) \cong \mathrm{Br}(T) \oplus \ker F$.
- It is not hard to identify $\ker F$ with the group $\mathscr{E}_G(T)$ of exterior equivalence class of $C_0(T)$-actions on $C_0(T, \mathcal{K})$.

$\mathscr{E}_G(T)$

- If $[C_0(T,\mathcal{K}),\alpha]$ is a $C_0(T)$-action in $\mathscr{E}_G(T)$, then evaluation at $t \in T$ gives an action α^t on $\mathcal{K}$.
- This gives us a well defined map $\varphi^\alpha : T \to H^2(G,\mathbf{T})$ where $\varphi^\alpha(t)$ is the Mackey obstruction $c(\alpha^t)$ for $\alpha^t : T \to \operatorname{Aut}\mathcal{K}$.
- It can be shown that φ^α is continuous and we get a map $\Phi : \mathscr{E}_G(T) \to C(T, H^2(G,\mathbf{T}))$.
- In general, $\ker\Phi$ consists of (exterior equivalence classes) of pointwise unitary automorphism groups and does not have an elegant description.
- However, under certain conditions, for example if $H^2(G,\mathbf{T})$ is Hausdorff and the abelianization $G_{\mathrm{ab}} = G/\overline{[G,G]}$ is compactly generated, Rosenberg has shown that $\ker\Phi$ consists of locally unitary automorphisms.
- Then work of Phillips & Raeburn shows that $\ker\Phi$ is parameterized by isomorphism classes of principal $\hat{G}_{\mathrm{ab}}$-bundles over T which are in turn parameterized by the sheaf cohomology group $H^1(T,\widehat{\mathcal{G}}_{\mathrm{ab}})$.

Getting There

Remark

A good description of $\mathscr{E}_G(T)$ requires both a description of the kernel of $\Phi : \mathscr{E}_G(T) \to C(T, H^2(G, \mathbf{T}))$ and a splitting map. Packer and later Packer & Raeburn showed Φ splits if G is an elementary abelian group ($G = \mathbf{R}^n \times \mathbf{T}^m \times \mathbf{Z}^k \times F$ where F is finite). Later Echterhoff and I extended this to "smooth" groups which include connected, simply connected Lie groups; connected semi-simple Lie groups; as well as all compact groups; discrete groups; and compactly generated abelian groups.

The Result

Theorem (EW)

Suppose that G is smooth as on the previous slide. Suppose that G acts trivially on T and that G_{ab} is compactly generated. Then

$$\mathrm{Br}_G(T) \cong \mathrm{Br}(T) \oplus C\big(T, H^2(G, \mathbf{T})\big) \oplus H^1(T, \widehat{\mathcal{G}}_{ab}).$$

Proof.

Because the action is trivial, the forgetful homomorphism is surjective and we always have $\mathrm{Br}_G(T) = \mathrm{Br}(T) \oplus \ker(F)$. We can identify $\ker F$ with $\mathscr{E}_G(T)$. If G is smooth, the map $\Phi : \mathscr{E}_G(T) \to C\big(T, H^2(G, \mathbf{T})\big)$ is surjective and splits. Then we can identify $\ker \Phi$ as on the previous slide. □

Break

Let's Take a Short Break

Remark

In the general case, the forgetful homomorphism $F : \mathrm{Br}_G(T) \to \mathrm{Br}(T) \cong H^3(T)$ will not be surjective. Nevertheless, the range is interesting not only to get a picture of the structure of $\mathrm{Br}_G(T)$, but because it allows us to determine which stable continuous-trace C^*-algebras with spectrum T admit actions covering the given action on T. As I will show, our analysis yields a very interesting and nontrivial answer to the later question.

The First Obstruction

Remark (Step One)

Note that $H^n(T)$ is a G-module. By functorality, $\mathrm{lt}_s \in \mathrm{Homeo}(T)$ induces an isomorphism $(\mathrm{lt}_s)^*$ and we let $s \cdot \mathfrak{c} = (\mathrm{lt}_s)^*(\mathfrak{c})$ for $\mathfrak{c} \in H^n(T)$. Thus if $\alpha \in \mathrm{Aut}\, A$ and A is a continuous-trace C^*-algebra with spectrum T, then α induces a homomoeomorphism α_* of T, and hence an isomorphism $(\alpha_*)^*$ of $H^3(T)$. An old result of Phillips & Raeburn asserts that $(\alpha_*)^*(\delta(A)) = \delta(A)$. This observation gives us our first obstruction for a class $\mathfrak{c} \in H^3(T)$ to lie in the range, $\mathrm{Im}\, F$, of the forgetful homomorphism.

Proposition (First Obstruction)

If $\mathfrak{c} \in \mathrm{Im}(F)$, then

$$\mathfrak{c} \in H^3(T)^G = \{\, \mathfrak{c} \in H^3(T) : s \cdot \mathfrak{c} = \mathfrak{c} \text{ for all } s \in G \,\}.$$

Second Obstruction

To further pin down $\operatorname{Im}(F)$, we need some more basic work of Phillips & Raeburn. Let $[A] \in \operatorname{Br}(T)$ and let $\operatorname{Aut}_{C_0(T)} A$ be the $C_0(T)$-linear automorphisms of A—which are necessarily locally inner. Then there is an obstruction $\zeta(\alpha)$ in $H^1(T, \mathcal{S}) \cong H^2(T)$ to α being inner. Then if A is stable, we have a short exact sequence

$$1 \longrightarrow \operatorname{Inn} A \longrightarrow \operatorname{Aut}_{C_0(T)} A \longrightarrow H^2(T) \longrightarrow 0$$

Let $\operatorname{Out} A = \operatorname{Aut}(A)/\operatorname{Inn} A$. Then we also get a short exact sequence

$$1 \to \operatorname{Aut}_{C_0(T)} A/\operatorname{Inn} A \to \operatorname{Out} A \to \operatorname{Homeo}_{\delta(A)}(T) \to 1$$

where $\operatorname{Homeo}_{\delta(A)}(T)$ are the homeomorphisms h of T such that $h^*(\delta(A)) = \delta(A)$. return return2

The Obstruction

Using the short exact sequences on the previous side, we can identify $\operatorname{Aut}_{C_0(T)} A/\operatorname{Inn} A$ with $H^2(T)$ and $\operatorname{Out} A/\operatorname{Im}(H^2(T)$ with $\operatorname{Homeo}_{\delta(A)}$ (see ses). If $\mathfrak{c} \in H^3(T)^G$, then there is an essentially unique stable A such that $\delta(A) = \mathfrak{c}$. But assumption on $\mathfrak{c}$, the given G-action on T gives us a homomorphism $\mathrm{It} : G \to \operatorname{Homeo}_{\delta(A)}(T)$. Thus we get a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H^2(T) & \longrightarrow & \operatorname{Out} A & \longrightarrow & \operatorname{Out} A/\operatorname{Im}(H^2(T)) & \longrightarrow & 1 \\ & & & & \lambda \uparrow & \nearrow_{\mathrm{It}} & & & \\ & & & & G & & & & \end{array}$$

If we can find an action covering It, then we can find λ such that the diagram communtes. General nonsense there is an obstruction to doing this in the group cohomology group $H^2(G, H^2(T))$. Making this work for the Moore groups requires some hypotheses.

The Second Obstruction

Lemma (Second Obstruction)

Suppose that $H^2(T)$ is countable. (For example, this is automatic if T has the homotopy type of a compact metric space.) Then there is a homomorphism $d_2 : H^3(T)^G \to H^2\big(G, H^2(T)\big)$ such that $d_2(\mathfrak{c}) = 0$ if and only if the map $\mathrm{lt} : G \to \mathrm{Homeo}_{\mathfrak{c}}(T)$ lifts to a continuous map $\lambda : G \to \mathrm{Out}(A)$ on the previous slide.

The Third Obstruction

We now suppose that $\mathfrak{c} \in H^3(T)^G$ and $d_2(\mathfrak{c}) = 0$. Then we want to lift λ so that

$$\begin{array}{ccc} & & \operatorname{Aut} A \\ & \overset{\alpha}{\nearrow} & \downarrow q \\ G & \xrightarrow[\lambda]{} & \operatorname{Out} A \end{array}$$

commutes for A with $\delta(A) = \mathfrak{c}$. Describing this obstruction a bit technical, so I will only crudely describe it here. Using λ, $C(X, \mathbf{T})$ becomes a G-module and we can form the Moore group $H^3(G, C(T, \mathbf{T}))$. Using Busby-Smith twisted actions and the "Packer-Raeburn Stabilization Trick", we can find an element $d_A(\lambda) \in H^3(G, C(T, \mathbf{T}))$ which vanishes when we can lift λ as above. Unfortunately, we have to worry about the choices of A and λ. It turns out that there is a homomorphism $d_2' : H^1(G, H^2(T)) \to H^3(G, C(T, \mathbf{T}))$ so that we obtain as well-defined homomorphism $d_3 : \ker d_2 \to H^3(G, C(T, \mathbf{T}))/ \operatorname{Im} d_2'$ so that if $\mathfrak{c} \in \operatorname{Im} F$, $d_3(\mathfrak{c}) = 0$.

The Theorem

Theorem (CKRW)

If $H^2(T)$ is countable, then the image of the forgetful homomorphism $F : \mathrm{Br}_G(T) \to H^3(T)$ consists of classes $\mathfrak{c} \in H^3(T)^G$ such that $d_2(\mathfrak{c}) = 0 = d_3(\mathfrak{c})$. In particular, a given stable continuous-trace C^*-algebra A with spectrum T admits a G-action covering the given action on T if and only if $\delta(A) \in H^3(T)^G$ and $d_2(\delta(A)) = d_3(\delta(A)) = 0$.

A Cool Corollary

Corollary

Suppose that T is a $\mathbf{R}$-space with $H^0(T)$, $H^1(T)$, and $H^2(T)$ all countable. If A is a stable continuous-trace C^*-algebra with spectrum T, then there is, up to exterior equivalence, exactly one action of $\alpha : \mathbf{R} \to \operatorname{Aut} A$ covering the given action on T.

Proof.

Since $\mathbf{R}$ is connected, It_0 and It_s are homotopic for all $s \in \mathbf{R}$ and $H^3(T) = H^3(T)^{\mathbf{R}}$. Since $H^2(T)$ is countable, Wigner has shown $H^2(\mathbf{R}, H^2(T))$ is trivial. Raeburn and Rosenberg have shown that $H^3(\mathbf{R}, C(T, \mathbf{T}))$ is trivial under our assumptions on $H^*(T)$. Hence all our obstructions in the previous theorem vanish and F is surjective. However $H^3(\mathbf{R}, C(T, \mathbf{T}))$ trivial also implies that there can be at most one (up to exterior equivalence) action. Hence the result. □

Remark

The corollary on the previous slide implies that $\ker F$ is trivial. In general, since F is a group homomorphism, $\ker F$ parameterizes (up to exterior equivalence) how many actions cover a given action on the spectrum of a continuous-trace C^*-algebra whenever that Dixmier-Douady class is in the range of F. Our next goal is to see what we can say in general about $\ker F$ and the structure of $\mathrm{Br}_G(T)$. It should be acknowledged that the motivation for all this—including our description of $\mathrm{Im}\, F$—comes from work of Kumjian and Parker. Kumjian in particular, used a spectral sequence of Grothendieck's which predicts many of the constructions here and below—even though no known generalization of Grothendieck's spectral sequence is known for non-discrete groups.

The Structure of $\mathrm{Br}_G(T)$

We are able to give a filtration $\{ e \} \subset B_1 \subset B_2 \subset \mathrm{Br}_G(T)$ and then describe the quotients $\mathrm{Br}_G(G)/B_2$ and B_2/B_1 in terms of Moore group cohomology groups. We will still need $H^2(T)$ countable throughout. A full discussion would take us too far afield, so we will settle for a crude summary here.

We let $B_2 = \ker F$. Thus B_2 is the subgroup $\mathscr{E}_G(T)$ of classes in $\mathrm{Br}_G(T)$ of the form $[C_0(T, \mathcal{K}), \alpha]$. We have already completely described

$$\mathrm{Br}_G(T)/\ker F = \mathrm{Im}\, F$$

as the subgroup of $H^3(T)^G$ determined by the homormorphisms d_2 and d_3.

The Homomorphism η

If $[C_0(T,\mathcal{K}),\alpha] \in \ker F$, then α induces the given action of G on T. Then for each $s \in G$, $\alpha_s \circ (\mathrm{lt}_s^{-1} \otimes 1) \in \mathrm{Aut}_{C_0(T)} C_0(T,\mathcal{K})$. The Phillips–Raeburn obstruction to this automorphism being inner is in $H^2(T)$ go and gives us a 1-cocycle from G to $H^2(T)$. Hence we get a homomorphism $\eta : \ker F \to H^1(G, H^2(T))$ whose range coincides with $\ker d_2' : H^1(G, H^2(T)) \to H^3(G, C(T,\mathbf{T}))$ we encountered earlier. Thus $B_2/B_1 = \ker F / \ker \eta$ is indentified with $\ker d_2' \subset H^1(G, H^2(T))$. We can also show that $B_1 = \ker \eta$ is isomorphic to a quotient of $H^2(G, C(T,\mathbf{T}))$.

That's all I want to say here about $\mathrm{Br}_G(T)$.

What About $S_G(T)$?

To get some information about $S_G(T)$ requires restriction to nice actions of G on T. First, we should recall what Rieffel-proper actions are. For simplicity we assume G is unimodular.

Definition

A dynamical system (A, G, α) is called Rieffel-proper if there is a dense α-invariant subalgebra A_0 of A such that

1. for all $a, b \in A_0$, $s \mapsto {}_\star\langle a \, , \, b\rangle(s) := a\alpha_s(b^*)$ is in $L^1(G, A)$,
2. For each $a, b \in A_0$ there is a unique α-invariant multiplier $\langle a \, , \, b\rangle_\star \in M(A)^\alpha$ such that for all $c \in A_0$ we have

$$\int_G c\alpha_s(a^*b)\,ds = c\,{}_\star\langle a \, , \, b\rangle.$$

The Result

We let $E_0 = \operatorname{span}\{\, {}_{\star}\langle a \,,\, b\rangle : a, b \in A_0 \,\}$. Then E_0 is an ideal in $A \rtimes_{\alpha,r} G$. We call the action saturated if E_0 is dense in $A \rtimes_{\alpha,r} G$. The closure A^{α} in $M(A)$ of $\operatorname{span}\{\, \langle a \,,\, b\rangle_{\star} : a, b \in A_0 \,\}$ is called the generalized fixed point algebra of α.

Theorem (Rieffel, Buss-Echterhoff)

Suppose that (A, G, α) is a saturated Rieffel-proper dynamical system. Then we can complete A_0 to a bimodule implementing a Morita equivalence between $A \rtimes_{\alpha,r} G$ and the generalized fixed point algebra A^{α}.

Free and Proper Actions

- If G acts freely and properly on T and if $(A, \alpha) \in \mathscr{S}_G(T)$, then α is Rieffel-proper and saturated.
- Moreover, in this case $A \rtimes_\alpha G = A \rtimes_{\alpha,r} G$, and $A \rtimes_\alpha G$ is Morita equivalent to A^α over $G \backslash T$.
- Thus we obtain a semigroup homomorphism $[A, \alpha] \mapsto [A^\alpha]$ from $S_G(T)$ to $S(G \backslash T)$.
- Even better, the pull-back $B \mapsto (C_0(T) \otimes_{C_0(G \backslash T)} B, \mathrm{lt} \otimes \mathrm{id})$ turns out to implement an inverse.
- Hence the following result due to an Huef, Raeburn, and myself.

$S_G(T)$ for Free and Proper Actions

Proposition (aHRW)

Suppose that G acts freely and properly on T. If $(A,\alpha) \in \mathscr{S}_G(T)$, then α is Rieffel-proper and saturated. Furthermore, A^α is Morita equivalent to $A \rtimes_\alpha G$ and the map $[A,\alpha] \mapsto [A^\alpha]$ is a semigroup isomorphism of $S_G(T)$ onto $S(G\backslash T)$.

Remark (Special Cases)

1. If $A = C_0(T)$, then the properness implies $G\backslash T$ is Hausdorff and then $A^\alpha = C_0(G\backslash T)$. Furthermore $C_0(T) \rtimes_{\mathrm{lt}} G$ is Morita equivalent to $C_0(G\backslash T)$.
2. If $A = C_0(T, D)$ and (D, G, α) is any dynamical system, then $(A, \gamma) = (C_0(T, D), \mathrm{lt} \otimes \alpha) \in \mathscr{S}_G(T)$ and $A^\gamma = \mathrm{Ind}_G^T(D, \alpha)$ where the later is the induced algebra consisting of continuous functions $f : T \to D$ such that $f(s^{-1} \cdot t) = \alpha_s\big(f(t)\big)$ and $G \cdot t \mapsto ||f(t)||$ vanishes at infinity.

- The proposition on the previous slide has an equivariant version.
- Suppose we have commuting free and proper actions of G (on the left) and H (on the right) of T.
- Suppose $(A, \alpha \times \beta) \in \mathscr{S}_{G\times H}(T)$.
- Then β extends to $M(A)$ and restricts to $\bar{\beta}$ on A^α, and $\bar{\beta}$ turns out to be continuous on A^α.
- Then $[A, \alpha \times \beta] \mapsto [A^\alpha, \bar{\beta}]$ gives an isomorphism of $S_{G\times H}(T)$ onto $S_H(G\backslash T)$.
- The situation is symmetric in G and H so we can prove the following.

Theorem (aHRW)

Suppose that we have commuting free and proper actions of G and H on T as above. Then there is a semigroup isomorphism

$$\theta : S_G(T/H) \to S_H(G\backslash T)$$

such that if $[B,\beta] = \theta\big([A,\alpha]\big)$, then $A \rtimes_\alpha G$ is Morita equivalent to $B \rtimes_\beta H$. Indeed, every class in $S_G(T/H)$ is of the form $[A^\beta, \bar\alpha]$ for some $(A, \alpha \times \beta) \in \mathscr{S}_{G\times H}(T)$, and we have

$$\theta\big([A^\beta, \bar\alpha]\big) = [A^\alpha, \bar\beta],$$

so that $A^\beta \rtimes_{\bar\alpha} G$ is Morita equivalent to $A^\alpha \rtimes_{\bar\beta} H$.

- Taking $A = C_0(T)$ gives the Morita equivalence between $C_0(G\backslash T) \rtimes_{\mathrm{rt}} H$ and $C_0(T/H) \rtimes_{\mathrm{lt}} G$ usually called Green's Imprimitivity Theorem.
- Since a semigroup isomorphism preserves the subgroups of invertible elements, we also obtain the following result due to Kumjian, Raeburn, and myself.

Theorem

Suppose G, H, and T are as above. Then θ restricts to a group isomorphism of $\mathrm{Br}_G(T/H)$ onto $\mathrm{Br}_H(G\backslash T)$.

The End

THANK YOU

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