Graded K-theory for $\mathbb{Z}_2$-graded graph C^*-algebras

Abend Seminar

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C^*-algebras

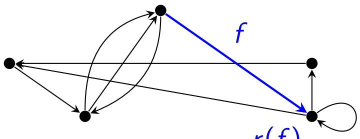
- Abstractly, a C^*-algebra is a complex Banach algebra with an involution such that $\|a^*a\| = \|a\|^2$ for all a.
- Examples:
  - $\mathbb{C}$;
  - $M_n(\mathbb{C})$;
  - $C(\mathbb{T})$, $C_0(\mathbb{R})$;
  - $C_0(X)$ where X is locally compact Hausdorff;
  - $B(\mathcal{H})$ with operator norm and adjoint $(T^*h|k) = (h|Tk)$;
  - $K(\mathcal{H}) \subseteq B(\mathcal{H})$
- Theorem (Gelfand–Naimark): every commutative C^*-algebra is $C_0(X)$ for some X.
- Theorem (Gelfand–Naimark): every C^*-algebra is a closed *-subalgebra of $B(\mathcal{H})$; orthogonal projections: $p = p^*p$; partial isometries: s^*s a projection.

K-theory for C^*-algebras

- Given a (unital) C^*-algebra A,
- $K_0(A)$ is the Grothendieck group of the semigroup $\{\text{projections in } \mathcal{K} \otimes A\}/(vv^* \sim v^*v)$ with $[p] + [q] = \left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right)$.
- $K_{i+1}(A)$ is $K_i(C_0(\mathbb{R}, A))$.
- Theorem: $K_1(A) \cong \{\text{unitaries in } \mathcal{K} \otimes A)^+\}/\text{homotopy}$.
- Theorem: $K_{i+2}(A) \cong K_i(A)$.

Graph C^*-algebras

Definition. A *directed graph* is $E = (E^0, E^1, r, s)$: E^0 is *vertex* set, E^1 is *edge* set, $r, s : E^1 \to E^0$ show directions.



Definition. Given $V \subseteq \{v \in E^0 : 0 < |s^{-1}(v)| < \infty\}$, define $C^*(E; V)$ as the C^*-algebra universal for mutually orthogonal projections $\{p_v \mid v \in E^0\}$ and elements $\{s_e : e \in E^1\}$ subject to
(CK1) $s_e^* s_e = p_{r(e)}$ for all e
(CK2) $p_v \geq \sum_{e \in F} s_e s_e^*$ for any finite $F \subseteq s^{-1}(v)$
(CK3) $p_v = \sum_{s(e)=v} s_e s_e^*$ for all $v \in V$.

Put $E^0_{\text{reg}} := \{v : 0 < |s^{-1}(v)| < \infty\}$; and $C^*(E) := C^*(E; E^0_{\text{reg}})$.

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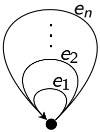
Example: the Cuntz algebras

For $n \geq 2$, let E be the graph



Relations for $C^*(E)$ boil down to $s_{e_i}^* s_{e_i} = 1 = \sum_j s_{e_j} s_{e_j}^*$.

This is the Cuntz algebra $\mathcal{O}_n$.

Note: If $n = 1$, then $C^*(E)$ is the universal C^*-algebra generated by a unitary: $C(\mathbb{T})$.

K-theory of $\mathcal{O}_n$

- (CK2) forces the $s_{e_i}s_{e_i}^*$ to be mutually orthogonal projections.
- Obvious projection in $\mathcal{O}_n$ is 1.
- Obvious relation as well:
  - $[1] = [s_v] = \sum_i [s_{e_i}s_{e_i}^*] \sim \sum_i [s_{e_i}^* s_{e_i}] = n \cdot [1]$.
  - So $(1-n)[1] = 0$.

Theorem (Cuntz): For $n \geq 2$, $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$; isomorphism takes $[1_{\mathcal{O}_n}]$ to $1 \in \mathbb{Z}$.

Note: if $n = 1$, still have $K_0(C(\mathbb{T})) \cong K^0(\mathbb{T}) = \mathbb{Z} \cong \mathbb{Z}/(1-1)\mathbb{Z}$.

K-theory for graph C^*-algebras

Recall:
$C^*(E) = \langle p_v, s_e \mid s_e^* s_e = p_{r(e)}, p_v = \sum_{s(e)=v} s_e s_e^* \text{ for } v \in E^0_{\mathrm{reg}} \rangle$.

The $[p_v]$ belong to $K_0(C^*(E))$ and there is a natural relation:
$[p_v] = \sum_{s(e)=v} [p_{r(e)}]$ for $v \in E^0_{\mathrm{reg}}$.

Write $A_E \in M_{E^0_{\mathrm{reg}}, E^0}(\mathbb{Z})$ for $A_E(v, w) = |s^{-1}(v) \cap r^{-1}(w)|$.

We obtain a map j from $\mathbb{Z}E^0 / (1 - A_E^t)\mathbb{Z}E^0_{\mathrm{reg}}$ to $K_0(C^*(E))$ carrying $\delta_v + (1 - A_E^t)\mathbb{Z}E^0_{\mathrm{reg}}$ to $[p_v]$.

Theorem (Pask–Raeburn, Raeburn–Szymański, Drinen–Tomforde). The map j is an isomorphism. Also, $K_1(C^*(E)) \cong \ker(1 - A_E^t)$.

Graded C^*-algebras

Definition. A $\mathbb{Z}_2$*-grading* of a C^*-algebra is an automorphism $\alpha : A \to A$ such that $\alpha^2 = \mathrm{id}_A$.

Put $A_0 = \{a \in A : \alpha(a) = a\}$ and $A_1 = \{a \in A : \alpha(a) = -a\}$.

Then $A_0 = \{a + \alpha(a) : a \in A\}$ and $A_1 = \{a - \alpha(a) : a \in A\}$, $A_i A_j \subseteq A_{i+j}$, and $A = A_0 \oplus A_1$.

Graded graph C^*-algebras

Fix a graph E and $V \subseteq E^0_{\mathrm{reg}}$.

Consider a labelling $\delta : E^1 \to \{0, 1\}$.

Recall that $\{p_v\}$ and $\{s_e\}$ are the generators of $C^*(E)$.

Consider $q_v := p_v$ $(v \in E^0)$ and $t_e = (-1)^{\delta(e)} s_e$ $(e \in E^1)$.

Since $t_e^* t_e = s_e^* s_e$ and $t_e t_e^* = s_e s_e^*$, these satisfy (CK1)–(CK3).

So universal property gives homomorphism $\alpha : C^*(E; V) \to C^*(E; V)$. And α^2 fixes generators so it's the identity.

So we have a grading.

Graded K-theory?

Numerous possible definitions of graded K-theory: *equivariant* K-theory; $K_*(A \rtimes_\alpha \mathbb{Z}_2)$; Karoubi; van Daele; Kasparov theory...

There is a graded tensor product operation $\widehat{\otimes}$ on graded C^*-algebras. Since $\mathcal{K}$ has a natural grading $\alpha_{\mathcal{K}}(\theta_{i,j}) = (-1)^{i-j}\theta_{i,j}$ we can take graded stabilisations $\mathcal{K} \mathbin{\widehat{\otimes}} A$.

So some properties we might expect:

- Homogeneous projections in $\mathcal{K} \mathbin{\widehat{\otimes}} A$ determine K_0-classes.
- Homogeneous partial isometries determine signed relations: $V^*V \sim \alpha(V)V^*$.

So we expect $K_0^{gr}(A)$ to look like equivalence classes of even projections in $A \mathbin{\widehat{\otimes}} \mathcal{K}$ modulo $[vv^*] = (-1)^j[v^*v]$ if $v \in (A \mathbin{\widehat{\otimes}} \mathcal{K})_j$.

Kasparov theory

- Working on the Novikov conjecture, Kasparov developed his KK-theory.
- Associates groups $KK_i(A, B)$ to pairs of graded C^*-algebras.
- Elements are equivalence classes of tuples (π, X, F, α_X);
  - X is a Hilbert B-module,
  - π is a left action of A on X,
  - α_X is a grading of X,
  - $F : X \to X$ anticommutes with α, and
  - $F^2 - 1$, $F - F^*$ and $F\pi(a) - \pi(a)F$ are all compact on X.
- Details are complicated, but a good illustrative example is $KK(\mathbb{C}, \mathbb{C})$:
  - a Kasparov $\mathbb{C}$–$\mathbb{C}$-module boils down to a Fredholm operator F on ℓ^2;
  - the corresponding class in $KK(\mathbb{C}, \mathbb{C})$ is determined by the Fredholm index of F.

Graded K-theory from Kasparov theory

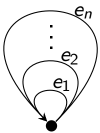
Theorem (Kasparov). For any trivially-graded C^*-algebra B, $KK_0(\mathbb{C}, B) \cong K_0(B)$.

This suggests $KK_0(\mathbb{C}, (A, \alpha))$ as a possible graded K_0-group.

Kasparov's work shows that a natural suspension operation for graded C^*-algebras is the graded tensor product with the first Clifford algebra $\mathbb{C}\mathrm{liff}_1 = \mathbb{C} \oplus \mathbb{C}$ graded by $\alpha_{\mathbb{C}\mathrm{liff}}(w, z) = (z, w)$.

We define $K_1^{gr}(A, \alpha) = K_0^{gr}(A \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{liff}\,1, \alpha \otimes \alpha_{\mathbb{C}\mathrm{liff}})$.

Haag's results



Haag studied $KK_*(\mathbb{C}, \mathcal{O}_n)$ for a grading α^δ corresponding to $\delta : \{e_1, \dots, e_n\} \to \{0, 1\}$:

Define $|\delta^{-1}(0)| = z$ and $|\delta^{-1}(1)| = o = n - z$. Haag proved $KK_0(\mathbb{C}, (\mathcal{O}_n, \delta)) \cong \mathbb{Z}_{1-z-o}$ and $K_1^{gr}(\mathcal{O}_n, \delta) = 0$.

So it "looks like" the ungraded calculation

$$[1] = [s_v] = \sum_i [s_{e_i} s_{e_i}^*] \sim \sum_i [s_{e_i}^* s_{e_i}] = n \cdot [1].$$

has become

$$[1] = [s_v] = \sum_i [s_{e_i} s_{e_i}^*] \sim \sum_i (-1)^{\delta(e_i)} [s_{e_i}^* s_{e_i}] = (z - o) \cdot [1].$$

This fits our "principles."

Pimsner–Voiculescu

- Consider a C^*-algebra A and an automorphism ρ of A.
- ρ carries projections to projections, so it induces a map $\rho_* : K_*(A) \to K_*(A)$.
- The *crossed product* $A \times_\rho \mathbb{Z}$ is generated by a copy $\pi(A)$ of A and a unitary U with $U\pi(a)U^* = \pi(\rho(a))$.
- the map π induces $\pi_* : K_*(A) \to K_*(A \times_\rho \mathbb{Z})$.

Theorem (Pimsner–Voiculescu). There are homomorphisms $K_i(A \times_\rho \mathbb{Z})$ that make the following sequence exact:

$$\begin{array}{ccccc}
K_0(A) & \xrightarrow{1-\rho_*} & K_0(A) & \xrightarrow{\pi_*} & K_0(A \times_\rho \mathbb{Z}) \\
\uparrow & & & & \downarrow \\
K_1(A \times_\rho \mathbb{Z}) & \xleftarrow{\pi_*} & K_1(A) & \xleftarrow{1-\rho_*} & K_1(A)
\end{array}$$

Graded Pimsner–Voiculescu sequence

Theorem (Kumjian–Pask–S). Let (A, α_A) be a graded C^*-algebra, and ρ a graded automorphism of A. For $j = \{0, 1\}$, there is a grading β^j of $A \times_\rho \mathbb{Z}$ such that $\beta^j(\pi(a)U^n) = (-1)^{nj}\pi(\alpha_A(a))U^n$. For $j = \{0, 1\}$ there are maps making the following sequence exact.

$$\begin{array}{ccccc}
KK_0(\mathbb{C}, A) & \xrightarrow{1-(-1)^j\rho_*} & KK_0(\mathbb{C}, A) & \xrightarrow{\pi_*} & KK_0(\mathbb{C}, A \times_\rho \mathbb{Z}) \\
\uparrow & & & & \downarrow \\
KK_1(\mathbb{C}, A \times_\rho \mathbb{Z}) & \xleftarrow{\pi_*} & KK_1(\mathbb{C}, A) & \xleftarrow{1-(-1)^j\rho_*} & KK_1(\mathbb{C}, A)
\end{array}$$

Definition. We take $KK_*(\mathbb{C}, A)$ as our definition of $K_*^{gr}(A)$.

Hilbert modules and their C^*-algebras

Hilbert bimodule X: like a Hilbert space, but scalars replaced by C^*-algebra A.

Given a Hilbert A–A module X, Pimsner constructed a C^*-algebra $\mathcal{O}_X$.

The algebra $\mathcal{O}_X$ is generated by a copy $\pi(A)$ of A and a copy $\psi(X)$ of X. So π induces $\pi_* : KK_*(B, A) \to KK_*(B, \mathcal{O}_A)$ for any B.

Examples:

- With $X = \mathbb{C}^n$ and $A = \mathbb{C}$, we have $\mathcal{O}_X = \mathcal{O}_n$.
- Given a graph E, there is a module X_E such that $\mathcal{O}_{X_E} \cong C^*(E)$.

Moreover, gradings α_X of X induce gradings $\alpha_{\mathcal{O}}$ of $\mathcal{O}_X$.

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Pimsner's exact sequences

Theorem (Pimsner). Take A, B trivially graded, and X a Hilbert A–A module. There are maps making the following sequence of groups exact:

$$\begin{array}{ccccc} KK_0(B,A) & \xrightarrow{1-[X]} & KK_0(B,A) & \longrightarrow & KK_0(B,\mathcal{O}_X) \\ \uparrow & & & & \downarrow \\ KK_1(B,\mathcal{O}_A) & \longleftarrow & KK_1(B,A) & \xleftarrow{1-[X]} & KK_1(B,A) \end{array}$$

Taking $X = X_E$ for a graph E, and $B = \mathbb{C}$, this gives back the Pask–Raeburn–Szymański description of $K_*(C^*(E))$.

Taking $X = {}_\rho A$, this gives the Pimsner–Voiculescu sequence.

Relative Cuntz–Pimsner algebras

Universal property of $\mathcal{O}_X$ involves relations on elements $a \in A$ that act compactly on X.

Muhly–Solel introduced *relative* Cuntz-Pimsner algebra $\mathcal{O}_{X,I}$: only impose relations for a in some ideal $I \lhd \mathcal{K}(X)$.

Fowler–Muhly–Raeburn: If $X = X(E)$, then taking $I = C_0(V)$ recovers $C^*(E; V)$.

Kumjian–Pask–S, Patterson–Sierakowski–S–Taylor: a labelling δ of E^1 gives a grading of $X(E)$, which induces a grading of $\mathcal{O}_{X(E),I}$, which agrees with α^δ.

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Pimsner sequence for graded KK-theory

Theorem (PSST). If (A, α_A) is a graded C^*-algebra and (X, α_X) a graded Hilbert A–A module, $I \lhd_{gr} A$ acts compactly on X, and (B, α_B) is any graded C^*-algebra, there are maps making the following sequence of groups exact:

$$\begin{array}{ccccc}
KK_0(B,I) & \xrightarrow{[\iota_I] - ([{}_I X_+] - [{}_I X_-])} & KK_0(B,A) & \longrightarrow & KK_0(B,\mathcal{O}_{X,I}) \\
\uparrow & & & & \downarrow \\
KK_1(B,\mathcal{O}_{X,I}) & \longleftarrow & KK_1(B,A) & \xleftarrow{[\iota_I] - ([{}_I X_+] - [{}_I X_-])} & KK_1(B,I)
\end{array}$$

Graded relative graph C^*-algebras

Corollary (PSST). Let E be a directed graph, V as subset of E^0_{reg} and $\delta : E^1 \to \mathbb{Z}_2$ a function. Let $A_V^\delta \in M_{V,E^0}(\mathbb{Z})$ be the matrix given by

$$A_V^\delta(v,w) = \sum_{s(e)=v \text{ and } r(e)=w} (-1)^{\delta(e)}.$$

Then

$$K_0^{gr}(C^*(E)) \cong \operatorname{coker}(1 - (A_V^\delta)^t) \qquad \text{and}$$
$$K_1^{gr}(C^*(E)) \cong \ker(1 - (A_V^\delta)^t)$$

There are similar formulas for the graded K-homology groups $KK(C^*(E), \mathbb{C})$.

So $K_0^{gr}(C^*(E;V))$ is roughly $\{\text{vertex projections}\}/\{[s_e^* s_e] = (-1)^{\delta(e)}[s_e s_e^*]\}$.

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