Simplicity of Nekrashevych algebras of contracting self-similar groups

Nóra Szakács
co-author: Benjamin Steinberg

University of York, UK
University of Szeged, Hungary

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Introduction

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- generalize Leavitt algerbras;
- Nekrashevych first introduced C^*-algebras, which generalize Cuntz algebras, and later studied their discrete counterparts;
- are examples of inverse semigroup algebras $\equiv$ ample groupoid (Steinberg) algebras.

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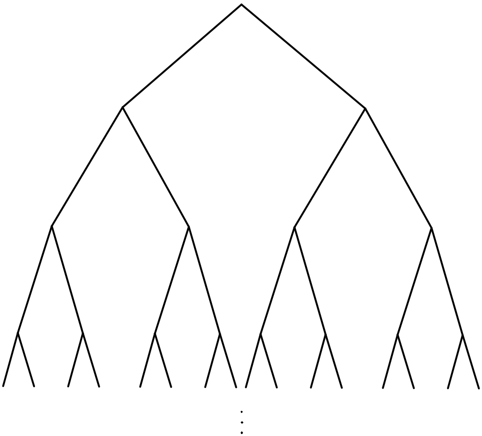
X: finite set with $|X| \geq 2$, T_X: $|X|$-regular tree



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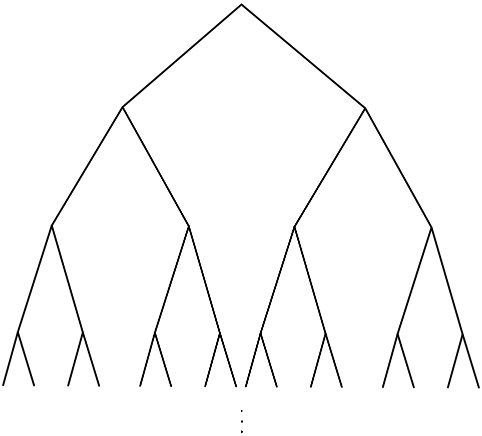

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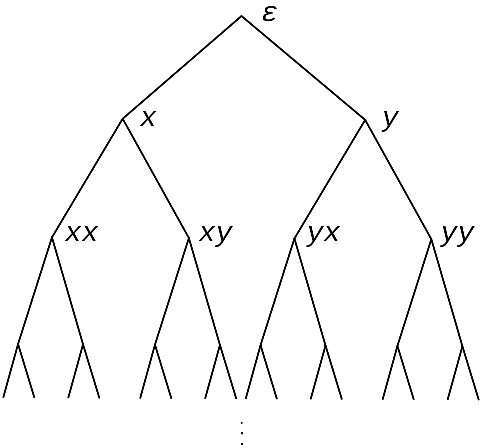

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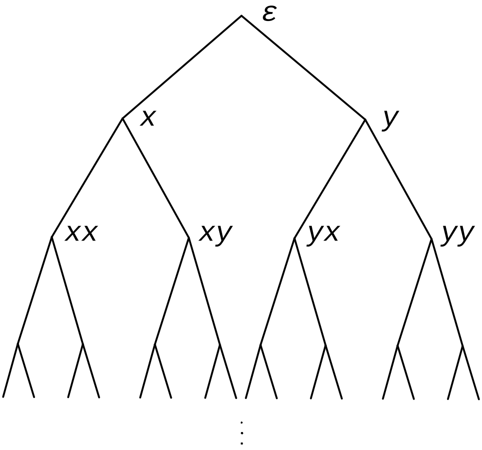

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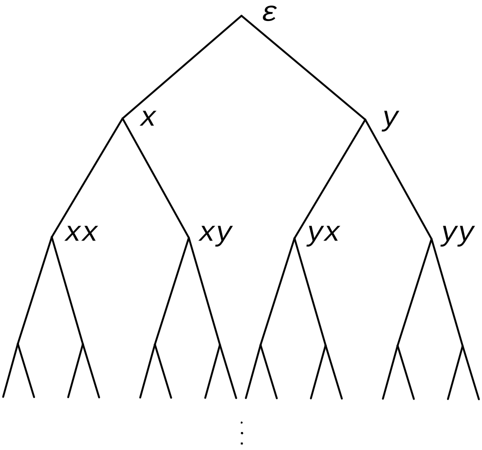

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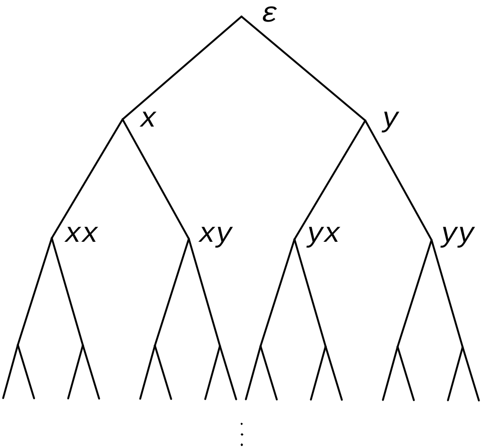

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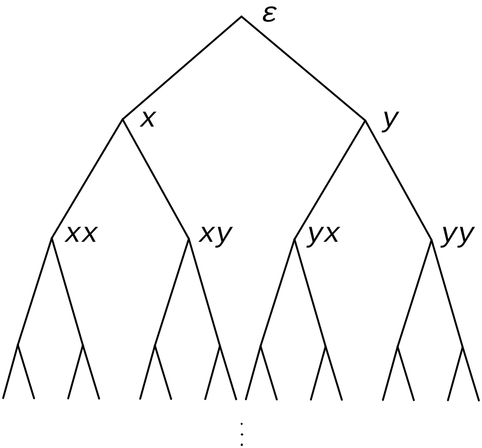

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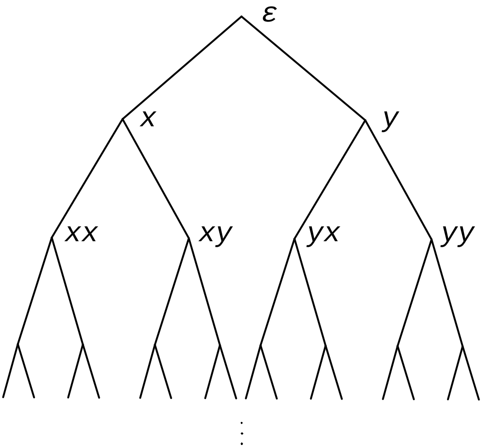

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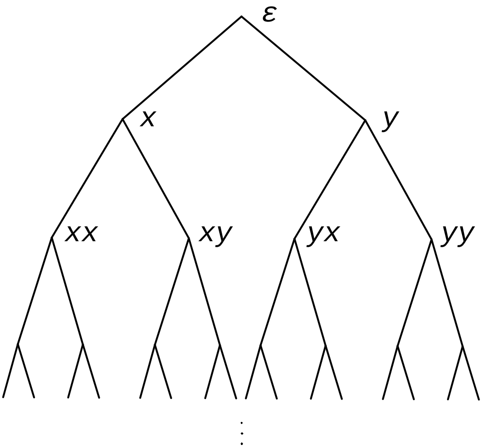

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- $g|_w \in \mathrm{Aut}(T_X)$: the restriction of g to the subtree at w
- $g(uw) = g(u)g|_u(w)$ for all $u, w \in X^*$

Self-similar groups

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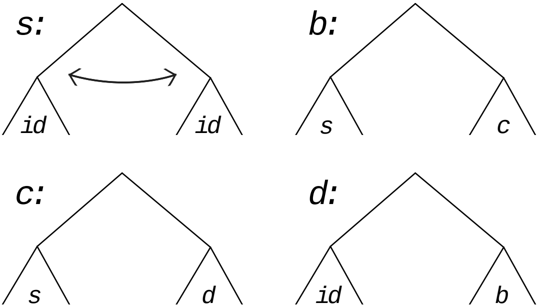
and were later linked to dynamical systems via iterated monodromy groups introduced by Nekrashevych.

The Grigorchuk group

The Grigorchuk group is generated by the following elements:



Contracting groups

A self-similar group $G \leq \mathrm{Aut}(T_X)$ is called **contracting** if there exists a finite set $N \subseteq G$ such that

for all $g \in G$ there exists k with $g|_w \in N$ whenever $|w| \geq k$.

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If G is contracting, and generated by a finite set closed under sections, then there is an algorithm that computes N.

Leavitt algebras

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(CK2) $\sum_{x \in X} xx^* = 1.$

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S: polycyclic monoid (inverse monoid)
$(1 - \sum_{x \in X} x x^*)$: the Cuntz-Krieger ideal, also coincides with the *tight ideal* $T_K(S)$ of S.

The representation on KX^ω

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so the above defines a representation of $L_K(X)$ on KX^ω, which is faithful.

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If G is contracting, G can be replaced by N, and $N_K(G,X)$ is finitely presented.

Nekrashevych algebras and inverse semigroups

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$N_K(G, X) = K_0S/(1 - xx^*)$, where again $(1 - xx^*)$ is the tight ideal $T_K(S)$ of S.

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- Nekrashevych (2019): the Nekrashevych algebra of the Grigorchuk-Erschler group is simple over no field

Simplicity of inverse semigroup algebras

Theorem (Steinberg, Sz.)

If S is a congruence-free inverse semigroup, then there is a unique maximal ideal of K_0S containing $T_K(S)$, called the ***singular ideal*** *$I_K(S)$.*

- $K_0S/T_K(S)$ has a unique maximal ideal: $I_K(S)/T_K(S)$.
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If G is self-similar over X, then the associated semigroup $S = \langle G, P_X \rangle$ is congruence-free. The singular ideal of K_0S is

$$I_K(S) = \{a \in K_0S : \text{for all } u \in X^* \text{ there is } w \in X^* \text{ with } auw = 0\}.$$

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So the unique maximal ideal of $N_K(G, X)$ is

$$I_K(S)/(1 - \sum_{x \in X} xx^*).$$

The unique simple quotient

Recall: G and $L_K(X)$ can be faithfully represented on KX^ω, and $N_K(G,X)$ too can be represented on KX^ω.

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Theorem (Steinberg, Sz.)

The unique maximal ideal of $N_K(G,X)$ is exactly the kernel of its representation on KX^ω.

The singular ideal minus the tight ideal

Let G be a self-similar over X and $S = \langle G, P_X \rangle$. We have established:

$$N_K(G, X) \text{ is simple} \iff I_K(S) \setminus (1 - \sum_{x \in X} xx^*) = \emptyset.$$

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Step 1:
For any $a \in K_0S$, we have $a \notin (1 - \sum_{x \in X} xx^*)$ if there are infinitely many words w with $aw \neq 0$.

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Step 2:
If $I_K(S) \setminus (1 - \sum_{x \in X} xx^*) \neq \emptyset$, then $I_K(S) \setminus (1 - \sum_{x \in X} xx^*)$ intersects KG. If G is contracting with nucleus N, it intersects KN.

Understanding $aw = 0$

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For any equivalence $\equiv$ on N, consider the following linear system in variables $x_g, g \in N$:

$$\sum_{g \equiv h} x_g = 0, \; h \in N.$$

We say $a \in KN$ satisfies $\equiv$ if $a_g = x_g$ is a solution over K.

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The key: to understand

- which of these must be satisfied for a to be singular,
- and which of these must not be satisfied for $a \notin (1 - \sum_{x \in X} xx^*)$.

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$$g \; x{\cdot}\equiv \; h \iff g(x) = h(x) \text{ and } g|_x \equiv h|_x.$$

The simplicity graph

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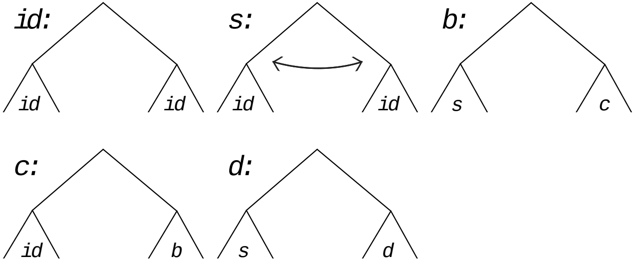
The simplicity graph: the Schreier graph of the action on $\{\equiv_w : w \in X^*\}$

An example: the Grigorchuk-Erschler group

The nucleus (also a generating set):



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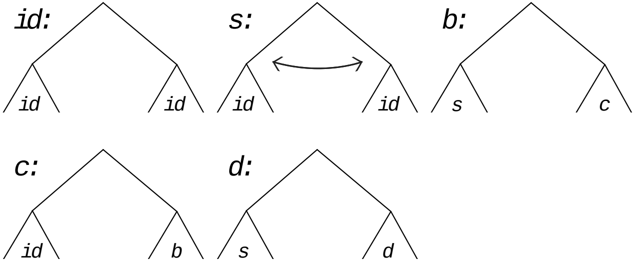
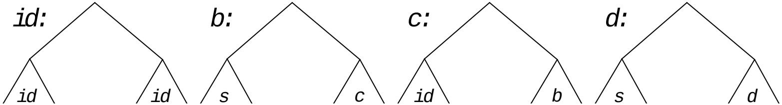
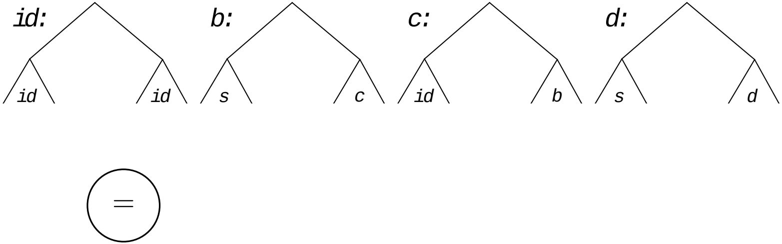
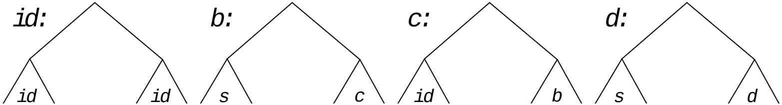
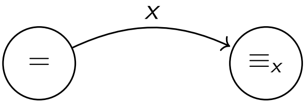


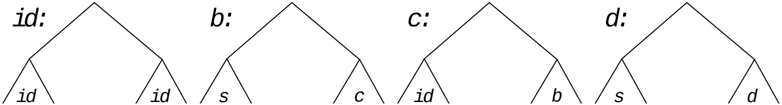
Note: $s(w) \neq g(w)$ if $s \neq g$, so s is its own $\equiv_w$-class for any w.

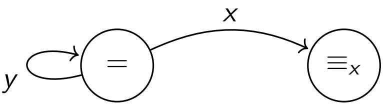

id:
id
id
b:
s
c
c:
id
b
d:
s
d

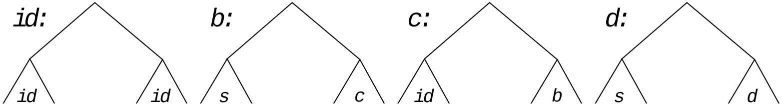

id:
id
id
b:
s
c
c:
id
b
d:
s
d
=

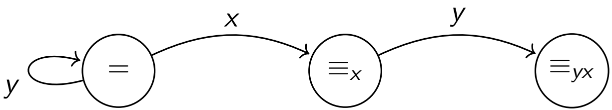

id:
id
id
b:
s
c
c:
id
b
d:
s
d

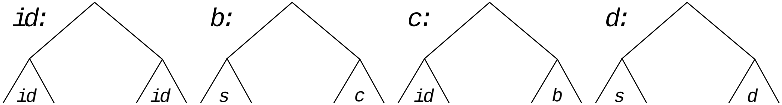

x
=
$\equiv_x$

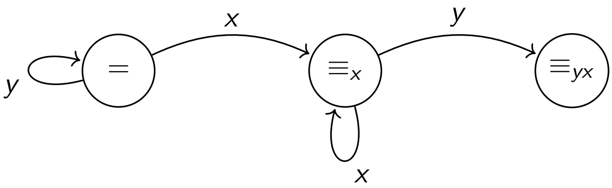

id:
id
id
b:
s
c
c:
id
b
d:
s
d

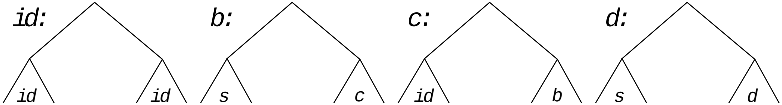

x
y
=
$\equiv_x$

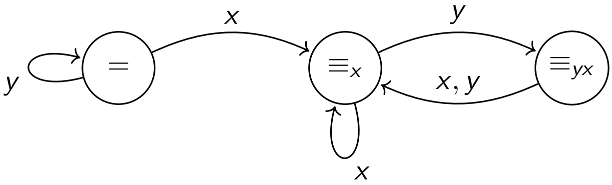

id:
id
id
b:
s
c
c:
id
b
d:
s
d


x
y
y
=
$\equiv_x$
$\equiv_{yx}$


id:
id
id
b:
s
c
c:
id
b
d:
s
d


y
=
x
≡x
x
y
≡yx


id:
id
id
b:
s
c
c:
id
b
d:
s
d


x
y
y
=
$\equiv_x$
$\equiv_{yx}$
x, y
x

The simplicity graph

For $u \in X^*$, let $\rho(u)$ be the reversal of u. Then there is a path from $\equiv_w$ to $\equiv_{uw}$ labeled by $\rho(u)$.

- $a \notin (1 - \sum_{x \in X} xx^*) \iff$ there are infinitely many words with $aw \neq 0$

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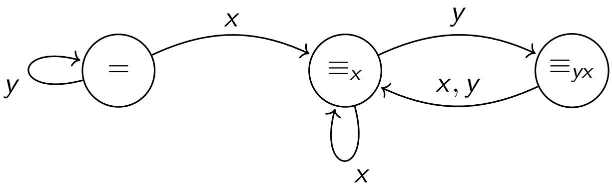
- a is singular $\iff$ for all $u \in X^*$ there is $w \in X^*$ with $auw = 0$ $\iff$ satisfies all the equations in the minimal strongly connected components.

The Grigorchuk-Erschler group



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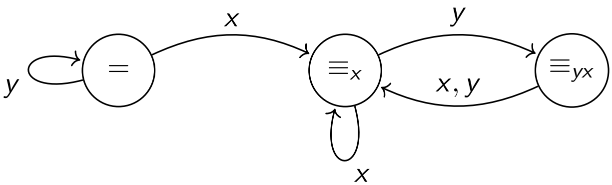

The minimal component:

$\equiv_x$: $\{s\}, \{b, d\}, \{id, c\}$

$\equiv_{yx}$: $\{s\}, \{c, d\}, \{id, b\}$

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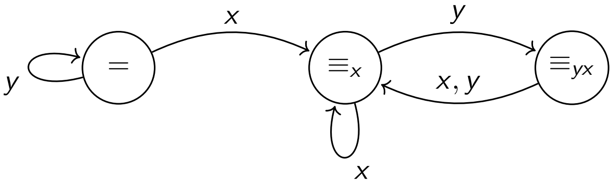

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$x_s = 0, x_d = -x_b, x_{\text{id}} = -x_c$
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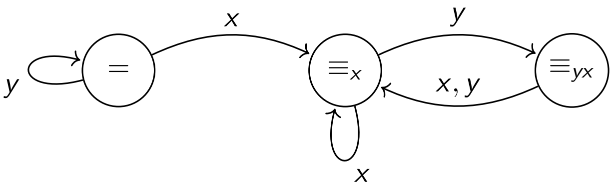

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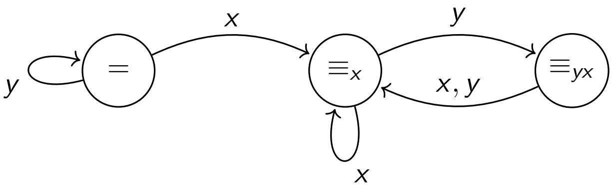

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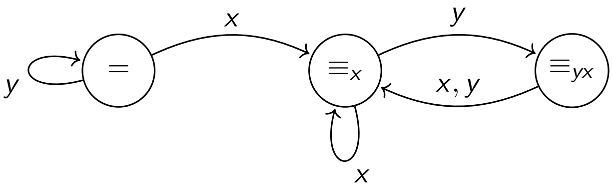

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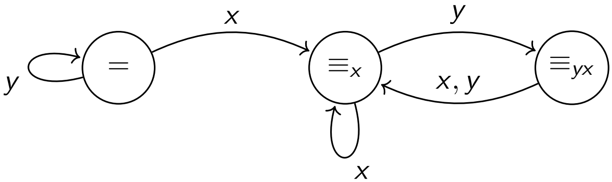

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$\Longrightarrow$ the Nekrashevych algebra is simple over no field.

Results

Theorem (Steinberg, Sz.)

Let G be a contracting group with nucleus N.

- *Either $N_K(G, S)$ is simple over no field or simple over all but finitely many positive characteristics.*
- *There is an algorithm which on input N outputs the set of characteristics over which $N_K(G, S)$ is non-simple.*

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Furthermore for any finite set of primes $\mathcal{P}$, we give a contracting self-similar group G such that $N_K(G,X)$ fails to be simple exactly over characteristics in $\mathcal{P}$.

Thanks for your attention!