Groupoid models and C^*-algebras of diagrams of groupoid correspondences

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An analogue of the space of complete histories

- Let $\mathcal{X}\colon \mathcal{G} \leftarrow \mathcal{G}$ be a groupoid correspondence.
- For $n \in \mathbb{N}$, form the n-fold composition $\mathcal{X}^{\circ n}\colon \mathcal{G} \leftarrow \mathcal{G}$.
- The map $\mathcal{X}^{\circ n+1} \to \mathcal{X}^{\circ n}$, $[(x_0, \dots, x_n)] \to [(x_0, \dots, x_{n-1})]$, is not well defined.
- $(x_0, \dots, x_n) \sim (x_0 g_0^{-1}, g_0 x_1 g_1^{-1}, \dots, g_{n-2} x_{n-1} g_{n-1}^{-1}, g_{n-1} x_n)$
- The map $\mathcal{X}^{\circ n+1}/\mathcal{G} \to \mathcal{X}^{\circ n}/\mathcal{G}$, $[(x_0, \dots, x_n)] \to [(x_0, \dots, x_{n-1})]$, is well defined.

Definition

Let $\Omega := \varprojlim \mathcal{X}^{\circ n}/\mathcal{G}$.

Lemma

There is a well defined action of $\mathcal{G}$ on Ω by $g \cdot [(x_0, x_1, x_2, \dots)] := [(g \cdot x_0, x_1, x_2, \dots)]$.
The map $\mathcal{X} \times_{\mathcal{G}^{(0)}} \Omega \to \Omega$, $(x_0, [(x_1, x_2, \dots)]) \mapsto [(x_0, x_1, x_2, \dots)]$, induces a canonical homeomorphism $\mathcal{X} \circ \Omega \xrightarrow{\sim} \Omega$.

The universal action

Theorem

The canonical homeomorphism $\mathcal{X} \circ \Omega \xrightarrow{\sim} \Omega$ is the universal action of the groupoid correspondence $\mathcal{X}$.

Proof.

- Let $\mathcal{X} \circ Y \xrightarrow{\sim} Y$ for a $\mathcal{G}$-space Y be any action of $\mathcal{X}$.
- Get canonical continuous $\mathcal{G}$-equivariant maps

$$Y \xrightarrow{\sim} \mathcal{X}^{\circ n} \circ Y := (\mathcal{X}^{\circ n} \times_{\mathcal{G}^{(0)}} Y)/\mathcal{G} \xrightarrow{\mathrm{pr}_1} (\mathcal{X}^{\circ n})/\mathcal{G}.$$

- They combine to a canonical continuous $\mathcal{G}$-map $Y \to \Omega$.
- This is the unique $\mathcal{X}$-equivariant map $Y \to \Omega$. □

The groupoid model

- The object space of the groupoid model must be the underlying space Ω of the universal $\mathcal{X}$-action.
- It remains to find the arrows for our groupoid model.
- We describe this through an inverse semigroup action.
- This is based on a set of bisections of $\mathcal{X}$.

Definition

Let $\mathcal{X}\colon \mathcal{G} \leftarrow \mathcal{G}$ be a groupoid correspondence.
A bisection of $\mathcal{X}$ is an open subset $U \subseteq \mathcal{X}$ such that $\sigma\colon \mathcal{X} \to \mathcal{G}^0$ and the quotient map $\mathcal{X} \to \mathcal{X}/\mathcal{G}$ are injective on U.

Lemma

The quotient map $\mathcal{X} \to \mathcal{X}/\mathcal{G}$ is a local homeomorphism.
The space $\mathcal{X}$ is covered by bisections.

How bisections act if $\mathcal{X}$ acts

- Let $\alpha\colon \mathcal{X} \circ Y \xrightarrow{\sim} Y$ be an action of $\mathcal{X}$ on Y.
- Let $U \subseteq \mathcal{X}$ be a bisection.
- $\varrho_Y^{-1}(\sigma_{\mathcal{X}}(U)) \subseteq Y$ is an open subset.
- Let $y \in \varrho_Y^{-1}(\sigma_{\mathcal{X}}(U)) \subseteq Y$.
- There is a unique $x \in U$ with $\sigma(x) = \varrho(y)$.
- Define $\vartheta_U(y) := \alpha(x, y) \in Y$.

Lemma

ϑ_U is a partial homeomorphism on Y.

Proof.

- Since α is a homeomorphism and the image of $U \times_{\mathcal{G}^{(0)}} Y$ in $\mathcal{X} \circ Y$ is open, the map ϑ_U is open.
- We must prove that it is injective.
- Let $\vartheta_U(y_1) = \vartheta_U(y_2)$.
- Let $x_1, x_2 \in U$ be unique with $\sigma(x_i) = \varrho(y_i)$ for $i = 1, 2$.
- $\alpha(x_1, y_1) = \alpha(x_2, y_2)$ ....

Proof continued

Lemma

ϑ_U is a partial homeomorphism on Y.

- $\alpha(x_1, y_1) = \alpha(x_2, y_2)$.
- $[(x_1, y_1)] = [(x_2, y_2)]$ in $\mathcal{X} \circ Y$.
- $(x_1, y_1) = (x_2 g^{-1}, g y_2)$ for some $g \in \mathcal{G}$.
- $[x_1] = [x_2]$ in $\mathcal{X}/\mathcal{G}$
- $x_1 = x_2$ in $\mathcal{X}$
- $g = 1$
- $y_1 = y_2$

Integrating an action of bisections?

Question

Let $\vartheta\colon \mathrm{Bis}(\mathcal{X}) \to \{\text{partial homeo's of } Y\}$.
When does ϑ come from an $\mathcal{X}$-action $\mathcal{X} \circ Y \xrightarrow{\sim} Y$?

- Let $(x, y) \in \mathcal{X} \times_{\sigma, \mathcal{G}^{(0)}, \varrho} Y$.
- There is $U \in \mathrm{Bis}(\mathcal{X})$ with $x \in U$.
- Is $\alpha[(x, y)] := \vartheta_U(y)$ a well-defined map $\mathcal{X} \circ Y \to Y$?
  Is α a homeomorphism?

The algebra of bisections

Definition

$V \in \mathrm{Bis}(\mathcal{G})$ acts on Y by a partial homeomorphism $\vartheta_V \colon \varrho^{-1}(s(V)) \to \varrho^{-1}(r(V))$, mapping $y \in \varrho^{-1}(s(V))$ to $g \cdot y$ for the unique $g \in V$ with $s(g) = \varrho(y)$.

Lemma

- *Let* $U, U_2 \in \mathrm{Bis}(\mathcal{X})$*,* $V \in \mathrm{Bis}(\mathcal{G})$*.*
- *Let* $\vartheta \colon \mathrm{Bis}(\mathcal{G}) \sqcup \mathrm{Bis}(\mathcal{X}) \to \{$*partial homeo's of* $Y\}$ *come from an* $\mathcal{X}$*-action.*
- $U \cdot V = \{x \cdot g : x \in U,\ g \in V\}$ *is a bisection of* $\mathcal{X}$*.*
- $V \cdot U = \{g \cdot x : x \in U,\ g \in V\}$ *is a bisection of* $\mathcal{X}$*.*
- $\vartheta_U \vartheta_V = \vartheta_{UV}$ *and* $\vartheta_V \vartheta_U = \vartheta_{VU}$*.*
- $\langle U \mid U_2 \rangle := \{g \in \mathcal{G} : Ug \cap U_2 \neq \emptyset\}$ *is a bisection of* $\mathcal{G}$*.*
- $\vartheta_U^* \vartheta_{U_2} = \vartheta_{\langle U \mid U_2 \rangle}$*.*

Lemma

- *Let* Y *be a* $\mathcal{G}$*-space.*
- *Let* $\vartheta\colon \operatorname{Bis}(\mathcal{G}) \to \{$*partial homeo's of* $Y\}$ *be the induced map.*
- *Let* $\vartheta\colon \operatorname{Bis}(\mathcal{X}) \to \{$*partial homeo's of* $Y\}$ *be a map.*
- *Assume* $\vartheta_U\vartheta_V = \vartheta_{UV}$ *and* $\vartheta_U^*\vartheta_{U_2} = \vartheta_{\langle U \mid U_2\rangle}$ *for* $U, U_2 \in \operatorname{Bis}(\mathcal{X})$, $V \in \operatorname{Bis}(\mathcal{G})$.
- *Then there is a well defined, injective, open, continuous map* $\alpha\colon \mathcal{X} \circ Y \hookrightarrow Y$ *with* $\alpha[(x, y)] = \vartheta_U(y)$ *for all* $(x, y) \in \mathcal{X} \times_{\mathcal{G}^{(0)}} Y$, $U \in \operatorname{Bis}(\mathcal{G})$, $x \in U$.
- *The map* α *is a homeomorphism* $\iff$ $\operatorname{im}\vartheta_U$ *for* $U \in \operatorname{Bis}(\mathcal{X})$ *cover* Y.

The groupoid model

Definition

Let S be the universal inverse semigroup generated by Θ_U for $U \in \mathrm{Bis}(\mathcal{G}) \sqcup \mathrm{Bis}(\mathcal{X})$ with the following relations:

- $\mathrm{Bis}(\mathcal{G}) \to S$ is a unital homomorphism;
- $\Theta_U \Theta_V = \Theta_{UV}$, $\Theta_V \Theta_U = \Theta_{VU}$ if $U \in \mathrm{Bis}(\mathcal{X})$, $V \in \mathrm{Bis}(\mathcal{G})$;
- $\Theta_{U_1}^* \Theta_{U_2} = \Theta_{\langle U_1 \mid U_2 \rangle}$ for $U_1, U_2 \in \mathrm{Bis}(\mathcal{X})$.

Theorem

- *Let Ω be the universal $\mathcal{X}$-action.*
- *The action on Ω induces an inverse semigroup action $\vartheta\colon S \to \{\text{partial homeo's of } \Omega\}$.*
- *$\Omega \rtimes_\vartheta S$ is the groupoid model for $\mathcal{X}\colon \mathcal{G} \leftarrow \mathcal{G}$.*

Sketch of the proof

- Let Y carry an $\mathcal{X}$-action.
- Since Ω is terminal, there is a unique $\mathcal{X}$-equivariant continuous map $f\colon Y \to \Omega$.
- The $\mathcal{X}$-actions on Y and Ω induce actions of S.
- The map f is S-equivariant.
- $\Omega \rtimes S$ acts on Y.
- The construction is natural: $\mathcal{X}$-equivariant maps are $\Omega \rtimes S$-equivariant.
- An action of $\Omega \rtimes S$ on Y induces an action of $\mathrm{Bis}(\mathcal{G})$ and an equivariant map to $\mathcal{G}^{(0)}$.
- This induces a $\mathcal{G}$-action on Y.
- $S \curvearrowright Y$ gives an injective, open, continuous map $\mathcal{X} \circ Y \hookrightarrow Y$.
- The analogous map $\mathcal{X} \circ \Omega \hookrightarrow \Omega$ is a homeomorphism.
- Since f is S-equivariant, $\mathcal{X} \circ Y \cong Y$.

The discrete case

- Let $\mathcal{G}$ be a discrete groupoid.
- Then $\mathcal{X}$ is a discrete set as well.
- $\Omega \rtimes S$ does not change when we replace $\mathrm{Bis}(\mathcal{G})$ and $\mathrm{Bis}(\mathcal{X})$ by the subsets of singleton or empty bisections.
- Thus we replace S by the inverse semigroup S' generated by $\mathcal{G} \sqcup \mathcal{X} \sqcup \emptyset$ with the following relations:
  - $\Theta_\emptyset$ is a zero element of S'.
  - Let $g, h \in \mathcal{G} \sqcup \mathcal{X}$ and not $g, h \in \mathcal{X}$. Then

$$\Theta_g \Theta_h = \begin{cases} \Theta_{g \cdot h} & \text{if } s(g) = r(h), \\ \Theta_\emptyset & \text{if } s(g) \neq r(h). \end{cases}$$

  - If $x \in \mathcal{X}$, then $\Theta_x^* \Theta_x = \Theta_{\sigma(x)}$ with $\sigma(x) \in \mathcal{G}^{(0)} \subseteq \mathcal{G}$.
  - If $x_1, x_2 \in \mathcal{X}$, $x_1\mathcal{G} \neq x_2\mathcal{G}$, then $\Theta_{x_1}^* \Theta_{x_2} = \Theta_\emptyset$.

Further relations in S'

- $\Theta_\emptyset = 0$
- $\Theta_g^* = \Theta_{g^{-1}}$ for $g \in \mathcal{G}$.
- There are well defined maps $\mathcal{X}^{\circ n} \to S'$, $[(x_1, \ldots, x_n)] \mapsto \Theta_{x_1} \cdots \Theta_{x_n}$.
- The map $\bigsqcup_{n \in \mathbb{N}} \mathcal{X}^{\circ n} \to S'$, $\omega \mapsto \Theta_\omega$, is "multiplicative":

$$\Theta_{\omega\eta} = \begin{cases} \Theta_\omega \Theta_\eta & \text{if } \sigma(\omega) = \varrho(\eta), \\ 0 & \text{otherwise.} \end{cases}$$

- Any non-zero element of S' is of the form $\Theta_\omega \Theta_\eta^*$ for $\omega, \eta \in \bigsqcup \mathcal{X}^{\circ m}$ with $\sigma(\eta) = \sigma(\omega)$.
- $\Theta_\omega^* \Theta_\omega = \Theta_{\sigma(\omega)}$ for $\omega \in \mathcal{X}^{\circ m}$.
- If $\omega, \omega' \in \mathcal{X}^{\circ m}$ and $[\omega] \neq [\omega']$ in $\mathcal{X}^{\circ m}/\mathcal{G}$, then $\Theta_\omega^* \Theta_{\omega'} = 0$.
- These relations give a formula for $\Theta_{\omega_1} \Theta_{\eta_1}^* \cdot \Theta_{\omega_2} \Theta_{\eta_2}^*$ for all $\omega_1, \eta_1, \omega_2, \eta_2 \in \bigsqcup \mathcal{X}^{\circ m}$:
  - Split the longer one of η_1 and ω_2 to get a combination $\Theta_{\eta_1'}^* \cdot \Theta_{\omega_2'}$ with η_1', ω_2' of the same length.
  - This is either zero or belongs to $\Theta(\mathcal{G}) = \Theta(\mathcal{G})^*$.
  - Then the product is zero or simplifies to a standard form.

The groupoid model and the Cuntz–Pimsner algebra

Theorem

Let $\mathcal{X} \colon \mathcal{G} \leftarrow \mathcal{G}$ be a proper groupoid correspondence.
Let $\mathcal{H}$ be its groupoid model.
Then $\mathrm{C}^*(\mathcal{H})$ is isomorphic to the absolute Cuntz–Pimsner algebra of $\mathrm{C}^*(\mathcal{X})$.

Some abstract nonsense

- $\mathcal{H} \curvearrowright \mathcal{H} \curvearrowleft \mathcal{H}$ induces $\mathcal{X} \curvearrowright \mathcal{H} \curvearrowleft \mathcal{H}$
  by the universal property of the groupoid model.
- The $\mathcal{X}$-action on $\mathcal{H}$ contains an action of $\mathcal{G}$.
- It commutes with the right $\mathcal{H}$-action.
- $\mathcal{H}$ becomes a groupoid correspondence $\mathcal{G} \leftarrow \mathcal{H}$.
- This correspondence $\mathcal{G} \leftarrow \mathcal{H}$ is proper:
  $\mathcal{H}/\mathcal{H} \cong \Omega$ and $\Omega \to \mathcal{G}^0$ is proper because $\mathcal{X}$ is proper.
- Get a proper C^*-correspondence $\mathrm{C}^*(\mathcal{H})\colon \mathrm{C}^*(\mathcal{G}) \leftarrow \mathrm{C}^*(\mathcal{H})$.
- This is a nondegenerate *-homomorphism $\mathrm{C}^*(\mathcal{G}) \to \mathrm{C}^*(\mathcal{H})$.

Composing C^*-correspondences

Definition

- Let A, B, C be C^*-algebras.
- Let $\mathcal{E}\colon A \leftarrow B$ and $\mathcal{F}\colon B \leftarrow C$ be C^*-correspondences.
- Equip $\mathcal{E} \otimes_{\mathrm{alg}} \mathcal{F}$ with the obvious A, C-bimodule structure and the inner product

$$\langle x_1 \otimes y_1 \mid x_2 \otimes y_2 \rangle := \langle y_1 \mid \langle x_1 \mid x_2 \rangle y_2 \rangle.$$

- This is positive definite.
- Completing $\mathcal{E} \otimes_{\mathrm{alg}} \mathcal{F}$ in it gives a C^*-correspondence $\mathcal{E} \circ \mathcal{F}\colon A \leftarrow C$.

Proposition

Let $\mathcal{G} \xleftarrow{\mathcal{X}} \mathcal{H} \xleftarrow{\mathcal{Y}} \mathcal{K}$ be groupoid correspondences.
Then $C^*(\mathcal{X} \circ \mathcal{Y}) \cong C^*(\mathcal{X}) \circ C^*(\mathcal{Y})$.

Representations of proper C*-correspondences

A, D C*-algebras

$\mathcal{E}$ proper A, A-correspondence

$\mathcal{F}$ A, D-correspondence with left action $\varphi \colon A \to \mathbb{B}(\mathcal{F})$

Definition (nondegenerate representation of $\mathcal{E}$ on $\mathcal{F}$)

isomorphism of A, D-correspondences $T \colon \mathcal{E} \circ \mathcal{F} \to \mathcal{F}$.

Theorem (Albandik–M 2016)

There is a natural bijection between representations of $\mathcal{E}$ on $\mathcal{F}$ and linear maps $S \colon \mathcal{E} \to \mathbb{B}(\mathcal{F})$ such that (φ, S) is a Cuntz–Pimsner covariant Toeplitz representation of $\mathcal{E}$:

$$S(x)(y) = T(x \otimes y)$$

Proof sketch for the theorem

- S is a Toeplitz representation
  $\iff$ T isometric and left A-linear.
- Remains: S Cuntz–Pimsner covariant $\iff$ T surjective.
- $\varphi_1(|x_1\rangle\langle x_2|)S(x_3) = S(x_1)S(x_2)^*S(x_3) = S(|x_1\rangle\langle x_2|x_3)$ for $x_1, x_2, x_3 \in \mathcal{E}$
- $\varphi_1(\psi(a))S(x_3) = S(a \cdot x_3) = \varphi(a)S(x_3)$ for $a \in A$, $x_3 \in \mathcal{E}$.
- Cuntz–Pimsner covariance holds on $S(\mathcal{E})\mathcal{F}$.
- T surjective $\iff$ $S(\mathcal{E})\mathcal{F} = \mathcal{F}$.
- Cuntz–Pimsner covariant $\implies$
  $\mathcal{F} \subseteq A \cdot \mathcal{F} \subseteq S(\mathcal{E})S(\mathcal{E})^*\mathcal{F} \subseteq S(\mathcal{E})\mathcal{F} \implies T$ surjective.

Cuntz–Pimsner algebra $\to$ groupoid C*-algebra

- Recall: $\mathcal{X} \curvearrowright \mathcal{H} \curvearrowleft \mathcal{H}$
- The isomorphism $\mathcal{X} \circ_{\mathcal{G}} \mathcal{H} \xrightarrow{\sim} \mathcal{H}$ induces

$$\mathrm{C}^*(\mathcal{X}) \circ \mathrm{C}^*(\mathcal{H}) \xrightarrow{\sim} \mathrm{C}^*(\mathcal{H})$$

- Since $\mathrm{C}^*(\mathcal{X})$ is a proper correspondence, this is equivalent to a Cuntz–Pimsner covariant Toeplitz representation $\mathrm{C}^*(\mathcal{X}) \to \mathrm{C}^*(\mathcal{H})$.
- It induces $\mathcal{O}_{\mathrm{C}^*(\mathcal{X})} \to \mathrm{C}^*(\mathcal{H})$.

Theorem

This *-homomorphism $\mathcal{O}_{\mathrm{C}^*(\mathcal{X})} \to \mathrm{C}^*(\mathcal{H})$ is an isomorphism.

The bicategories of groupoid and C*-correspondences

- Groupoid correspondences and C*-correspondences are not the arrows of a category because their composition is only associative up to canonical isomorphisms.
- Such situations also occur in other contexts.
- The concept of a bicategory formalises this.
- A monoid action in a bicategory has extra data for the compatibility with multiplication.
- A monoid action in the C*-correspondence bicategory is a product system over the monoid.
- A diagram of groupoid correspondences induces a product system.
- The diagram also has a groupoid model.
- Sometimes the C*-algebra of the groupoid model is isomorphic to the Cuntz–Pimsner algebra of the product system – but not always.

Composition is weakly unital and associative

- The composition of groupoid correspondences is associative and unital up to natural isomorphisms

Definition

Let A be a C*-algebra.
Then A with the obvious Hilbert A-module structure and left multiplication by A is a correspondence $A \leftarrow A$, called the unit correspondence.

Lemma

For any correspondence $\mathcal{E}\colon A \leftarrow B$,

$$\begin{aligned} A \circ \mathcal{E} &\cong \mathcal{E}, & \mathcal{E} \circ B &\cong \mathcal{E}, \\ a \otimes x &\mapsto a \cdot x, & x \otimes b &\mapsto x \cdot b. \end{aligned}$$

If $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ are composable correspondences, then

$$(\mathcal{E} \circ \mathcal{F}) \circ \mathcal{G} \cong \mathcal{E} \circ (\mathcal{F} \circ \mathcal{G}),$$
$$(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z).$$

The bicategory of C*-correspondences

There is a bicategory that has

objects	C*-algebras
arrows $B \to A$	C*-correspondences $\mathcal{E}\colon A \leftarrow B$
2-arrows $\mathcal{E}_1 \Rightarrow \mathcal{E}_2$	linear maps $\mathcal{E}_1 \to \mathcal{E}_2$ that are A, B-bimodule maps and isometric for the B-valued inner products ("correspondence maps")
product	$\circ$
vertical product	if $\mathcal{E}_i\colon A \leftarrow B$ for $i = 1, 2, 3$ and $S\colon \mathcal{E}_1 \Rightarrow \mathcal{E}_2$ and $T\colon \mathcal{E}_2 \Rightarrow \mathcal{E}_3$, then the vertical product of T and S is $T \circ S\colon \mathcal{E}_1 \Rightarrow \mathcal{E}_3$
horizontal product	if $\mathcal{E}_1, \mathcal{E}_2\colon A \leftarrow B$ and $\mathcal{F}_1, \mathcal{F}_2\colon B \leftarrow C$, and $S\colon \mathcal{E}_1 \Rightarrow \mathcal{E}_2$, $T\colon \mathcal{F}_1 \Rightarrow \mathcal{F}_2$, then their horizontal product is $S \otimes_B T\colon \mathcal{E}_1 \otimes_B \mathcal{F}_1 \Rightarrow \mathcal{E}_2 \otimes_B \mathcal{F}_2$

Associators and unitors are the canonical isomorphisms $(\mathcal{E} \circ \mathcal{F}) \circ \mathcal{G} \cong \mathcal{E} \circ (\mathcal{F} \circ \mathcal{G})$, $A \circ \mathcal{E} \cong \mathcal{E}$, and $\mathcal{E} \circ B \cong \mathcal{E}$.

Diagrams in a bicategory

- Let M be a monoid, $\mathcal{C}$ a bicategory.
- A homomorphism $M \to \mathcal{C}$ consists of
  - an object $A \in \mathcal{C}$;
  - arrows $\alpha_g \colon A \to A$ for $g \in M$;
  - invertible 2-arrows $\mu_{g,h} \colon \alpha_g \alpha_h \Rightarrow \alpha_{gh}$ for $(g,h) \in M$

  such that $\alpha_1 = 1_A$, $\mu_{1,h}$ and $\mu_{g,1}$ are the canonical 2-arrows, and the following diagrams commute for $(g,h,k) \in M^3$:

$$\begin{array}{ccccc} \alpha_g \alpha_{hk} & \xLeftarrow{1 \bullet \mu_{h,k}} & \alpha_g(\alpha_h \alpha_k) \cong (\alpha_g \alpha_h)\alpha_k & \xRightarrow{\mu_{g,h} \bullet 1} & \alpha_{gh}\alpha_k \\ & \searrow^{\mu_{g,hk}} & & \swarrow^{\mu_{gh,k}} & \\ & & \alpha_{ghk} & & \end{array}$$

- A homomorphism from M to the C*-correspondence bicategory is the same as a product system over M.
- Diagrams in a bicategory are even a bit more general, allowing M to be a (bi)category.

Diagrams of groupoid correspondences

- A diagram of groupoid correspondences over a monoid M consists of

$\mathcal{G}$	groupoid
$\mathcal{X}_g$	groupoid correspondences for $g \in M$
$\mu_{g,h}$	isomorphisms of groupoid correspondences $\mathcal{X}_g \circ \mathcal{X}_h \xrightarrow{\sim} \mathcal{X}_{gh}$

  such that
  - $\mathcal{X}_1 = \mathcal{G}$
  - $\mu_{1,h} \colon \mathcal{G} \circ \mathcal{X}_h \xrightarrow{\sim} \mathcal{X}_h$ and $\mu_{g,1} \colon \mathcal{X}_g \circ \mathcal{G} \xrightarrow{\sim} \mathcal{X}_g$ are the canonical isomorphisms
  - μ is “associative”
- Applying C^* turns this into a diagram of C^*-correspondences.
- This is a product system, and it has a Cuntz–Pimsner algebra.

Link to higher-rank graphs

Proposition

Let V be a topological space viewed as a groupoid.
A rank-k topological graph with vertex space V is equivalent to a diagram of groupoid correspondences with $M = \mathbb{N}^k$ and $\mathcal{G} = V$.

Actions of diagrams of groupoid correspondences

- Take a diagram of groupoid correspondences over M:

  $\mathcal{G}$ groupoid
  $\mathcal{X}_g$ groupoid correspondences for $g \in M$
  $\mu_{g,h}$ isomorphisms of groupoid correspondences
  $\mathcal{X}_g \circ \mathcal{X}_h \xrightarrow{\sim} \mathcal{X}_{gh}$

- An action of the diagram on a locally compact space Y consists of
  - a $\mathcal{G}$-action on Y
  - $\mathcal{G}$-equivariant homeomorphisms $\alpha_g \colon \mathcal{X}_g \circ Y \xrightarrow{\sim} Y$ for $g \in G$

  such that
  - $\alpha_1 \colon \mathcal{G} \circ Y \xrightarrow{\sim} Y$ is the canonical homeomorphism
  - the following diagrams commute for all $g, h \in M$

$$\mathcal{X}_g \circ Y \xLeftarrow{1 \bullet \alpha_h} \mathcal{X}_g \circ (\mathcal{X}_h \circ Y) \cong (\mathcal{X}_g \circ \mathcal{X}_h) \circ Y \xRightarrow{\mu_{g,h} \bullet 1} \mathcal{X}_{gh} \circ Y$$

$$\mathcal{X}_g \circ Y \xRightarrow{\alpha_g} Y \xLeftarrow{\alpha_{gh}} \mathcal{X}_{gh} \circ Y$$

The groupoid model of a diagram

- The groupoid model of a diagram of proper groupoid correspondences is defined as for a single groupoid correspondence:
  its actions on locally compact spaces are in natural bijection with actions of the diagram
- The groupoid model exists. It is unique up to isomorphism.
- Its object space is the underlying space of the universal action of the diagram.
- The Cuntz–Pimsner algebra of the product system associated to the diagram maps canonically to the groupoid C^*-algebra of the groupoid model.
- This map is always surjective, but not always injective.

Theorem

Let M be an Ore monoid such as $M = \mathbb{N}^k$.
Then the Cuntz–Pimsner algebra of the product system associated to a diagram of proper groupoid correspondences is isomorphic to the groupoid C^*-algebra of the groupoid model.