Graded K-theory, filtered K-theory and the classification of Leavitt path algebras

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Definition of Leavitt path algebras. Examples.

Definition

Let $E = (E^0, E^1, r, s)$ be a directed graph (where $r\colon E^1 \to E^0$ is the range map and $s\colon E^1 \to E^0$ is the source map). Let k be a field.

The **Leavitt path algebra** $L_k(E)$ is the k-algebra given by generators $\{v, e, e^* \mid v \in E^0,\ e \in E^1\}$, subject to the following relations:

(V) $vw = \delta_{v,w} v$ and $v = v^*$ for all $v, w \in E^0$,

(E) $s(e)e = er(e) = e$ for all $e \in E^1$,

(CK1) $e^* f = \delta_{e,f} r(e)$ for all $e, f \in E^1$, and

(CK2) $v = \sum_{e\in E^1 : s(e)=v} ee^*$ whenever $0 < |s^{-1}(\{v\})| < \infty$.

Examples

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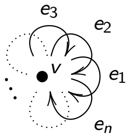
We have $L_k(A_n) \cong M_n(k)$.

The *n*-rose quiver R_n:



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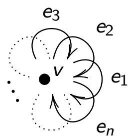

We have that $L_k(R_n) \cong L_k(1, n) =$

$$= \langle X_1, \dots, X_n, X_1^*, \dots, X_n^* \mid X_i^* X_j = \delta_{ij} 1, \sum_{i=1}^{n} X_i X_i^* = 1 \rangle$$

is the *Leavitt algebra* of type $(1, n)$, i.e., $L \cong L^n$.

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Definition

$L_k(E)$ is a graded algebra over $\mathbb{Z}$:

$$L_k(E) = \bigoplus_{n\in\mathbb{Z}} L_k(E)_n,$$

where $L_k(E)_n$ is spanned by elements $\gamma\nu^*$ with $|\gamma| - |\nu| = n$.

The involution

Given any involution $*$ on k, there is a unique involution on $L_k(E)$ such that

$$(r\lambda\nu^*)^* = r^*\nu\lambda^*.$$

Note that $L_k(E)_n^* = L_k(E)_{-n}$ for $n \in \mathbb{Z}$.

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Remark

When $k = \mathbb{C}$, the Leavitt path algebra $L_{\mathbb{C}}(E)$ embeds as a dense $*$-subalgebra of $C^*(E)$, the graph C^*-algebra of E.

We denote by $K_0^{gr}(L_k(E))$ the graded K_0-group of $L_k(E)$. It is the group of differences $[P] - [Q]$, where P and Q are *graded* f.g. projective $L_k(E)$-modules.

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- $K_0^{gr}(L_k(E))$ is also an ordered group, with positive cone

$$K_0^{gr}(L_k(E))^+=\{[P]:P \text{ graded f.g. proj. module }\}.$$

Conjecture

(Hazrat's Conjecture) Let E and F be two finite graphs and let k be a field. Assume there is a $\mathbb{Z}$-equivariant isomorphism of ordered groups

$$K_0^{gr}(L_k(E)) \cong K_0^{gr}(L_k(F))$$

sending $[L_k(E)]$ to $[L_k(F)]$. Then $L_k(E) \cong L_k(F)$ as graded k-algebras.

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Conjecture

(Hazrat's Conjecture II) Let E and F be arbitrary graphs and let k be a field. Assume there is a $\mathbb{Z}$-equivariant isomorphism of ordered groups

$$K_0^{gr}(L_k(E)) \cong K_0^{gr}(L_k(F)).$$

Then $L_k(E)$ and $L_k(F)$ are graded Morita-equivalent.

Let E a finite essential graph, meaning that E has neither sources nor sinks. Then we can write

$$L_k(E) = L_k(E)_0[t_+, t_-; \alpha],$$

as graded $*$-algebras, where $\alpha\colon L_k(E)_0 \to eL_k(E)_0e$ is a corner-isomorphism of $L_k(E)_0$.

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If $\alpha\colon R \to eRe$ is an isomorphism, where R is a unital k-algebra, the algebra $R[t_+, t_-; \alpha]$ is the algebra generated by R, t_+, t_-, with the relations $t_-t_+ = 1$ and $t_+rt_- = \alpha(r)$ for $r \in R$.

Theorem (A-Pardo, 2014)

Let k be a field and let E and F be finite essential graphs. Suppose that there is a $\mathbb{Z}$-equivariant isomorphism of ordered groups

$$K_0^{gr}(L_R(E)) \cong K_0^{gr}(L_R(F))$$

sending $[L_k(E)]$ to $[L_k(F)]$. Then there exists a locally inner automorphism g of $L_k(E)_0$ such that

$$L_k^g(E) \cong_{gr} L_k(F),$$

where $L_k^g(E) = L_k(E)_0[t_+, t_-; \alpha \circ g]$.

Filtered K-theory (Eilers, Restorff, Ruiz, Sørensen)

Definition

Let R be a graded ring. For $k \leq 0 \leq m$, the *filtered K-theory* $\mathrm{FK}_{k,m}(R)$ in the range $[k, m]$ is the collection of algebraic K-theory groups

$$K_n(I),\ K_n(I/J), \qquad (k \leq n \leq m)$$

for graded ideals $J \subset I$, together with all the exact sequences

$$K_n(I_2/I_1) \xrightarrow{\iota_*} K_n(I_3/I_1) \xrightarrow{\pi_*} K_n(I_3/I_2) \xrightarrow{\partial_*} K_{n-1}(I_2/I_1)$$

for graded ideals $I_1 \subset I_2 \subset I_3$ and $k < n \leq m$.

Main result

Theorem (A-Hazrat-Li)

Let E, F be row-finite graphs and k a field. Suppose that there exists an order-preserving $\mathbb{Z}$-equivariant isomorphism

$$\varphi\colon K_0^{gr}(L_k(E)) \longrightarrow K_0^{gr}(L_k(F)).$$

Then the lattices of graded ideals of $L_k(E)$ and $L_k(F)$ are isomorphic and moreover

$$\mathrm{F}\overline{\mathrm{K}}_{0,1}(L_k(E)) \cong \mathrm{F}\overline{\mathrm{K}}_{0,1}(L_k(F)),$$

where $\mathrm{F}\overline{\mathrm{K}}_{0,1}(L(E))$ is a certain quotient of $\mathsf{FK}_{0,1}(L_k(E))$ that we will describe later.

The talented monoid

A-Moreno-Pardo introduced the *graph monoid* M_E for a row-finite graph E:

$$M_E = \langle E^0 : v = \sum_{e \in s^{-1}(v)} r(e) \rangle.$$

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They showed that

$$\mathcal{V}(L_k(E)) \cong M_E \cong \mathcal{V}(C^*(E)),$$

where $\mathcal{V}(A)$ is the monoid of isomorphism classes of f.g. projective modules over A.

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$$M_E^{gr} = \left\langle v(i), v \in E^0, i \in \mathbb{Z} : v(i) = \sum_{e \in s^{-1}(v)} r(e)(i-1) \right\rangle$$

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associated to a row-finite graph E.

Theorem (A-Hazrat-Li-Sims)

For a row-finite graph E we have

(a) $M_E^{gr} \cong \mathcal{V}^{gr}(L_k(E))$*, and*

(b) M_E^{gr} *is a cancellative monoid, and thus it is isomorphic to* $K_0^{gr}(L_k(E))^+$.

It turns out that the talented monoid M_E^{gr} contains a lot of information about $L_k(E)$. For instance, it contains full information on its lattice $\mathcal{L}^{gr}(L_k(E))$ of graded ideals:

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Here $\mathcal{T}_E$ is the lattice of hereditary saturated subsets of E^0.

An exact sequence in K-theory

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We establish an exact sequence:

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where $\phi[v(i)] = [v(i+1)] - [v(i)]$, and where $U([v(i)]) = [v]$ forgets the grading.

The (algebraic) K_1-group $K_1(L_k(E))$ of $L_k(E)$ was computed by A-Brustenga-Cortiñas.

Let E be a row-finite graph. Let A_E be the adjacency matrix of E, with the zero rows (corresponding to the sinks of E) suppressed. Let I be the matrix obtained from the identity $E^0 \times E^0$ matrix suppressing the columns corresponding to the sinks. We consider the matrix $A_E^t - I$ as a homomorphism

$$A_E^t - I \colon \mathbb{Z}^R \to \mathbb{Z}^{E^0},$$

where $R = E^0 \setminus \mathrm{Sink}(E)$ are the non-sinks of E and $\mathbb{Z}^X$ indicates the free abelian $\mathbb{Z}$-module on X for each set X.

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Indeed, the matrix $A_E^t - I$ induces a homomorphism

$$A_E^t - I\colon G^R \to G^{E^0}$$

for every abelian group G.

There is a non-canonical splitting

$$K_1(L_k(E)) \cong \operatorname{coker}\Big(A_E^t - I\colon (k^\times)^R \to (k^\times)^{E^0}\Big) \bigoplus \ker\Big(A_E^t - I\colon \mathbb{Z}^R \to \mathbb{Z}^{E^0}\Big)$$

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where $k^\times = k \setminus \{0\}$ is the multiplicative group of the units of k.

In the case of graph C^*-algebras we have an isomorphism

$$K_1^{\text{top}}(C^*(E)) \cong \ker\Big(A_E^t - I \colon \mathbb{Z}^R \to \mathbb{Z}^{E^0}\Big),$$

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$$K_1^{\mathrm{top}}(C^*(E)) \cong \ker\left(A_E^t - I \colon \mathbb{Z}^R \to \mathbb{Z}^{E^0}\right),$$

and there is a formula, obtained by Carlsen, Eilers and Tomforde, giving the isomorphism

$$\chi \colon \ker\left(A_E^t - I \colon \mathbb{Z}^R \to \mathbb{Z}^{E^0}\right) \to K_1^{\mathrm{top}}(C^*(E)).$$

The formula for χ can be adapted to the case of Leavitt path algebras and gives a map

$$\chi' \colon \ker\Big(A_E^t - I \colon \mathbb{Z}^R \to \mathbb{Z}^{E^0}\Big) \to K_1(L_k(E)).$$

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However this map χ' is not a group homomorphism. The problem is that the map χ' depends on the choice of certain permutations.

Theorem

Let G_E be the subgroup of $K_1(L_k(E))$ generated by $[(-v)+(1-v)]_1$, for all $v \in E^0$. Then there is a well-defined group homomorphism

$$\chi\colon ker\Big(A_E^t - I\colon \mathbb{Z}^R \to \mathbb{Z}^{E^0}\Big) \to K_1(L_k(E))/G_E.$$

Moreover this map is functorial with respect to the maps induced by graded subquotients of $L_k(E)$, and it is a section of a canonical homomorphism $\xi\colon K_1(L_k(E))/G_E \to ker\Big(A_E^t - I\colon \mathbb{Z}^R \to \mathbb{Z}^{E^0}\Big)$ in algebraic K-theory (an index map).

We can now define the quotient $\mathrm{F}\overline{\mathrm{K}}$ of filtered K-theory.

Definition

Let E be a row-finite graph and k be a field. Define the *algebraic filtered $\overline{K}$-theory* $\mathrm{F}\overline{\mathrm{K}}_{0,1}(L_k(E))$ as the collection

$$\{\overline{K}_n(J/I)\}_{0\leq n\leq 1}$$

where (I,J) ranges over all the graded ideals of $L_k(E)$ such that $I\subseteq J$,

$$\overline{K}_0(J/I)=K_0(J/I)$$

and

$$\overline{K}_1(J/I)=K_1(J/I)/G_{J/I}$$

where the subgroup $G_{J/I}$ of $K_1(J/I)$ is defined by using that J/I is canonically a Leavitt path algebra of a suitable graph.

Shift equivalence

Let A, B be square matrices with coefficients in $\mathbb{Z}^+$.

We say that A and B are *shift equivalent* if there are matrices R, S over $\mathbb{Z}^+$ and a positive integer ℓ such that

$$SR = A^\ell, \quad RS = B^\ell, \quad BR = RA, \quad AS = SB.$$

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If E and F are finite graphs without sinks, we say that E and F are shift equivalent if their adjacency matrices A_E and A_F are shift equivalent.

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It can be shown that E and F are shift equivalent if and only if $K_0^{gr}(LE)$ and $K_0^{gr}(LF)$ are equivariantly order-isomorphic.

Theorem (A-Hazrat-Li)

Let E and F be finite graphs without sinks. If E and F are shift equivalent, then the graph C^*-algebras $C^*(E)$ and $C^*(F)$ are Morita equivalent.

The proof uses:

Theorem (Eilers, Restorff, Ruiz, Sørensen)

Let E and F be graphs. If there is an isomorphism

$$FK_{0,1}(L_{\mathbb{C}}(E)) \cong FK_{0,1}(L_{\mathbb{C}}(F)),$$

then there is an isomorphism

$$FK^{top}(C^*(E)) \cong FK^{top}(C^*(F)).$$

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This result can be easily adapted to show that the same holds replacing $\mathrm{FK}_{0,1}$ with $\mathrm{F\overline{K}}_{0,1}$.

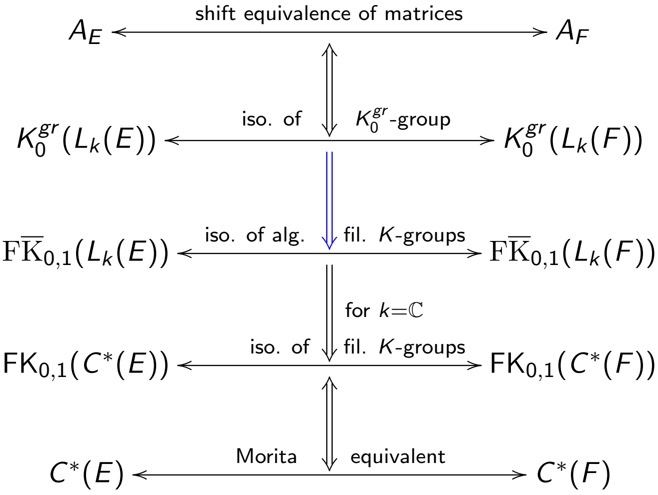
The proof:
Let E and F be finite graphs without sinks.

If E and F are shift equivalent, then $K_0^{gr}(LE) \cong_{\mathbb{Z}} K_0^{gr}(LF)$.

By our main theorem, we then have $\mathrm{F}\overline{\mathrm{K}}_{0,1}(LE) \cong \mathrm{F}\overline{\mathrm{K}}_{0,1}(LF)$.

By [ERRS], $\mathsf{FK}^{\mathsf{top}}(C^*(E)) \cong \mathsf{FK}^{\mathsf{top}}(C^*(F))$

By a main result of ERRS, $C^*(E) \sim_M C^*(F)$.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Thank you very much for your attention!!!

The simplest example in which χ_1' depends on the choice of bijections is when E has only one vertex v and one edge e. In this case, $L_k(E) = k[t, t^{-1}]$ and $K_1(L_K(E)) = K_0(k) \oplus K_1(k) = \mathbb{Z} \oplus k^\times$. Consider $x = 2 \in \mathrm{Ker}(A_E^t - I) = \mathbb{Z}$. Then $L_x^+ = \{(v,1),(v,2)\}$ and $L_x^- = \{(e,1),(e,2)\}$. Now taking $[v,i] = i$ and $\langle e,i\rangle = i$ for $i = 1,2$, we obtain $\chi_1'(2) = \left[\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \right]_1$. Now taking $[v,i] = i$ for $i = 1,2$ and $\langle e,1\rangle' = 2$, $\langle e,2\rangle' = 1$, we get $\tilde{\chi}_1'(2) = \left[\begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix} \right]_1 \neq \chi_1'(2)$ (if the characteristic of k is different from 2)