Teaching students maths so they learn:

constructing mental models in mathematics

- **Part 1: Mental models in experts and novices**
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Introduction

This paper looks at the sort of pictures, or “mental models” that students construct in their heads when they are learning mathematics.

This paper purely focuses on the types of mental models constructed as students learn mathematics. Clearly, this is only one component of student learning. This paper does not discuss other learning theory, nor the neuroscience, nor the psychology relating to student learning, all of which are also important considerations in helping students to learn mathematics effectively.

In Part 1, this paper looks at the concept of a mental model, and compares the mental models of students to those of experts. In Part 2, it looks at what can be done to assist students to develop high quality mental models in mathematics.

The information in this paper has been sourced from the literature on learning mathematics. As the body of literature on constructing mental models in university students studying mathematics is currently small, some work studying mental models in college physics students and children studying mathematics has also been considered here.

Part 1: Mental models in experts and novices

What is a mental model?

When you see or hear a word, for example, “dog”, your mind springs to action and provides you with a whole lot of associations.

You may see an image of a dog.

You may hear the dog barking or panting.

You may smell the dog.

You may see a mental movie of the dog doing something: maybe chasing a ball or playing.

You may recall the feel of the dog’s fur.

The word “dog” may arouse emotional associations – good ones if you have a pet dog that you love; or bad ones if you don’t like dogs.

A whole lot of other, more abstract associations will also spring to mind: mammal, four-legged, loyal companion, and whatever else you associate with dogs.

This complete set of ideas and associations form your mental model for “dog”. The exact nature of your mental model is personalised to you and is dependent on many things, including your experiences, they way your brain interprets sensory inputs, and your education.

Mental models in mathematics

We also form mental models when we are learning mathematics.

As mathematicians and maths educators, our mathematical mental models are not dissimilar to our models for things like "dog": they are accurate, detailed, highly interconnected, and useful.

For example:

The word "integers" may conjure up thoughts of:

- …, -3, -2, -1, 0, 1, 2, 3, …;
- one-to-one correspondences;
- the number line;
- induction;
- $\aleph_0$;
- **Z**; subsets of **R**;
- rings; and so on.

When **we** learn maths, we, consciously or unconsciously, form some sort of a mental model that lets us synthesize and understand all the information – all the words, pictures, formulae, relationships, and so on – in a way that's consistent and makes sense. As **our** mathematical understanding develops, our models move from concrete to abstract and from simple to sophisticated.

Consequently, we have a good understanding of maths, and enjoy doing maths.

Not so for most of our students.

We are different from most of our students

"It seems like the groundswell of people who hate, fear or are frustrated by math are in the majority."

Margaret Paton, 2018

This includes the majority of our students.

It is very important for us to understand that we are different from most of our students.

Therefore, we should not assume that they think or learn in the same way as us.

We need to know how they think, and what they are thinking in order to teach them to develop good mental models.

The next sections discuss what we know about the way students think when they are learning maths. The final sections discuss what we know about teaching to develop good mental models in students.

Student maths mental models

Researchers have reported that:

- Most students never develop mental models for the maths they are taught at uni. They rely on memorized procedures.
- Of those who do develop mental models, most of their models are:
  - Not good enough;
  - Contain contradictions;
  - Wrong; or
  - Not useful.

Yet, we know that you can teach people to develop good mental models because it is being done successfully in other fields.

Doing maths without good mental models

To experience what it is like to do maths without good mental models, try the following experiments.

1. Make $1.70 using only 10c, 20c and 50c coins. (Assume you have unlimited supplies of each type of coin.)

This is very easy because you have a mental model in your head for how money works. It is associated with your model for place value, and your model for addition, and your model for the number line (to measure how close you are to $1.70).

2. Now make $1.58 using only 7c, 12c and 20c coins.

This is difficult, because you don’t have a model in your head that lets you get to the value.

To solve this, you probably either painstakingly worked out a pattern that gets you to the value you need, or you used trial and error.

If you don’t have a pattern or model, then you are forced to use trial-and-error on each and every case, and then memorize the combinations to get to each value.

This is what it is like to do maths without an operational mental model. It is what many students are faced with when they do maths, and it makes doing maths prohibitively difficult.

Six differences between mental models in experts and novices

There is a continuum of mental models for any given topic, ranging from the models of the most proficient expert, through to the models of the weakest student. In order to present the following information in the clearest manner, each difference will be presented as a dichotomy between experts and novices. This is not to say, by any means, that all students present as novices in the way that is described here. In many instances, strong students will have models that are more closely aligned to those of the expert, but weak students are more likely to have mental models more closely aligned to those described here as belonging to the novice.

Appreciating these differences can inform our teaching and allow us to develop methods to promote the development of good mental models in students.

1. Memorized procedures

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Some (most) novices have no mental models for the maths they are learning, and rely on memorized procedures.

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For example:

Differentiation quiz

Q1. Find $\lim_{x \to 2} (\frac{x^2 - 5x + 6}{x - 2})$ if it exists

Student's answer: *"Use the quotient rule …"*

You can see that this student has memorized a procedure that says, "if I am doing differentiation and I see a fraction, I need to apply the quotient rule."

Having no mental models and relying on memorized procedures is the worst case.

Many such students do not even realise that there is more to maths than memorizing formulae and algorithms.

They may not be aware that maths is actually a cohesive whole and they should have a mental model that joins the different formulas and concepts together and explains how they work.

When you teach something to these students, they look for a procedure or formula to memorize, because this is how their mental framework is set up. They don't store the understanding or the big picture or the underlying process, because there is no place for these in their mental framework.

These students get through exams by learning to use cues in questions to recall the correct formula or procedure. Part of their procedure is to find trigger words or variables in questions that indicate the formula to use. They get through exams essentially without any understanding.

You could teach the student from this example that actually this is a limit problem, so you need to do the following … but this is just replacing one procedure with another. It's not adding any understanding. We need to look at ways to change this student's mental framework.

2. Choice of problem features

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Novices choose what to do based on surface features of a problem. Experts choose models based on the structural features of a problem.

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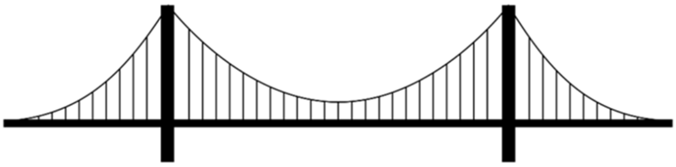
In other words, novices view the problem as a set of independent details and they choose one or two of them to focus on, whereas experts see the problem as a single whole.

For example:

(Exam question)

The Golden Gate bridge in San Francisco is a large suspension bridge.



The cable joining the two towers of the bridge (see illustration) can be modeled by the equation:

$$y = \frac{3}{8000}x^2 - \frac{9}{20}x + 155$$

where y is the height of the cable above the bridge deck in metres and x is the horizontal distance from the left tower in metres.

What is the distance between the towers? (Hint: both towers are the same height. The left-hand tower is at $x = 0$)

Student solution:

$$x = \frac{\frac{9}{20} \pm \sqrt{\left(\frac{9}{20}\right)^2 - 4 \times \frac{3}{8000} \times 155}}{2 \times \frac{3}{8000}}$$

$x = 1221$ or $x = -21.8$

You can see exactly what's happened. The students see the quadratic equation and immediately go and find its roots (just like they've been drilled to do at school), without looking at the problem in its entirety (and realising that y ≠ 0!).

3. Level of detail

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Experts and novices have different levels of detail in their models.

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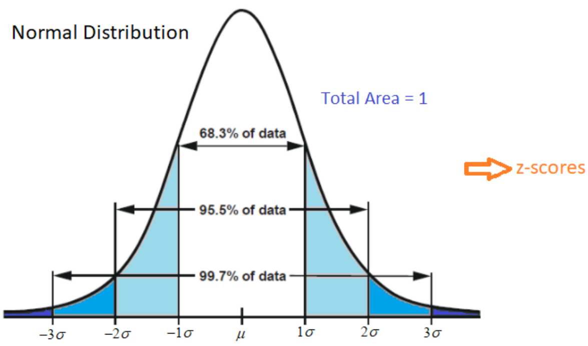
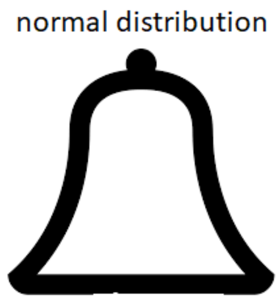
Strong students make more detailed mental models. Expert mental representations demonstrate superior extent, depth and level of detail They accommodate more interconnections and are more geared towards action.

Weak students, on the other hand, have simpler, fuzzier models with less interconnections that are less useful.



Expert model



Novice model

Expert and novice models for the normal distribution.

Note that this is only an illustration and does not reflect actual mental models!

4. Contradictory models

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Novices comfortably hold contradictory models.

Experts do not.

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For example:

In a study, children were individually shown the fraction sum $\frac{2}{7}+\frac{1}{7}$. Most children evaluated it to $\frac{3}{14}$. The researcher then provided the children with Cuisenaire rods, and they constructed the sum and evaluated it to $\frac{3}{7}$. When asked to explain this, some of the children explained, “when you do it in algebra you get $\frac{3}{14}$ but when you do it with rods, the answer is $\frac{3}{7}$”. (The other answers fell into 3 groups: the algebra is wrong; the rods are wrong; and “I don’t know”.)

Our students can comfortably hold contradictions.

This is because students’ models are purely functional – they only need to get to the right answer.

When they meet a new fact, they simply tack it onto what they already know, whether it fits or not.

So they can hold contradictory models and learn which part of the model applies in which situation without ever having to address the contradiction.

If students are not put in situations where they are forced to confront the contradictory models, the contradictions will most likely remain present and unaddressed.

5. Model adaptation

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When given new information, experts adapt their models to integrate it.

Novices either just add it onto their model, or they learn it as a new, independent situation (regardless of possible contradictions).

As per item 4., if students are not put in situations where they are forced to confront the contradictory models, the contradictions will most likely remain present and unaddressed.

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For example:

The children who decided that $\frac{2}{7}+\frac{1}{7}$ has different answers in different contexts were simply adding the new information to their mental model without attempting to integrate it consistently with their existing knowledge.

6. Model choice

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Experts hold multiple models and select the best model for a problem. Novices do not!

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Think back to your models of the integers; if the only way you conceived the integers was in a one-to-one correspondence with pieces of fruit, you would struggle to do problems (such as working with negative numbers) that require different models.

For example:

"Imagine a wire-frame cube resting on a tabletop with the front face directly in front of you and perpendicular to your line of sight. Imagine the long diagonal that goes from the bottom, front, left-hand comer to the top, back, right-hand one. Now imagine that the cube is reoriented so that this diagonal is vertical and the cube is resting on one comer. Place one fingertip about a foot above a tabletop and let this mark the position of the top corner on the diagonal. The corner on which the cube is resting is on the tabletop, vertically below your fingertip. With your other hand point to the spatial locations of the other corners of the cube."

Hinton, G. (1979). Some Demonstrations of the Effects of Structural Descriptions in Mental Imagery. *Cognitive Science*, *3*, 231–250.

Geoffrey Hinton performed this experiment on over 20 people, only one of whom gave the correct answer.

When presented with this problem most people show the locations of 4 additional vertices, forgetting that the cube has 8 vertices. It appears that rotating all the parts of a cube is too difficult.

In Hinton's study, the person who got the answer correct rotated both the axes and the cube, a model that allowed the problem to be solved easily.

Weak students tend to have a single model for an item that they inflexibly try to use in all situations.

Part 2: Teaching students how to develop good mental models

There is clear evidence that mathematical experts hold good mathematical mental models, but that few students of mathematics do.

We want students to develop good mental models, yet most of them fail to do so.

What can we do to change this?

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There is a strong case for explicitly teaching students how to store maths in their brains as well as what to store.

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How do we go about this?

Building mental models is hard

Firstly, we need to appreciate that building good mental models is hard, and hard work, and in mathematics, it does not generally seem to happen automatically.

Davis (1983) wrote that,

> *"Neither the importance of mental representations, nor the difficulty in creating them, seem to be generally recognized."*

Constructing good mental models requires time, effort and guidance.

Conceptual models

A conceptual model is a visual representation that facilitates comprehension. In maths, these are typically diagrams.

We routinely use such diagrams to illustrate concepts, and to succinctly synthesize a set of information in a way that shows the connections between the components.

Many teachers expect that their use of instructive conceptual models will allow students to learn and develop an understanding of the concepts under instruction.

> *"When we teach, it is common to assume that students have acquired or constructed mental models that are copies of the conceptual models [diagrams] that have been presented to them."*
>
> Greca, I. & Moreira, M. (2000). Mental models, conceptual models, and modelling. *International Journal of Science Education*, 22(1), 1–11.

However,

> *"Regardless of the effort that teachers invest on their task … students limit themselves to learning by heart long lists of formulae and definitions, which they do not understand, because the phenomena described there are not being interpreted according to the mental models the students should be constructing. "*
>
> Greca, I. & Moreira, M. (1997). The kinds of mental representations-models, propositions and images-used by college physics students regarding the concept of field. *International Journal of Science Education*, 19(6), 711–724.

Teaching with conceptual models is excellent practice, but is not sufficient for the development of good mental models. If it was, then all of our students would naturally develop good mental models, which they unfortunately don't.

> *"It seems reasonable that there should be a simple and direct relation between a conceptual model and a mental model, but this does not seem to be the case."*
>
> Norman, D. 1983. "Some observations on mental models". In *Mental models*, Edited by: Centnerend, D. and Stevens, A. 6–14. N.J.: Lawrence Erlbeum Associetes (p12).

This is a somewhat unexpected result.

Moreover, it is not only students whose mental models don't match up with conceptual models.

The mental models of active physicists have been found not to match the conceptual (textbook) models either.

But, when these same scientists present their findings, they present them via conceptual diagrams and equations (just like in a textbook) but they don't disclose the process that they used to get to these conceptions.

Typically, students never see the internal mental models of the scientists, (or mathematicians in our case) and they never see how to construct good mental models (from conceptual models, or otherwise). Students are meant to acquire this skill implicitly. A minority of students somehow manage to achieve this - these are the students who become capable mathematics students.

From concrete … to conceptual … to mental model

We know that nearly all students have access to good processing capabilities when dealing with concrete materials.

We also know that students are typically poor at making connections between the concrete and symbolic representations. Thus, we should explicitly model the process of developing conceptual models from the concrete.

We know that good mental representations are very difficult for students to construct.

The link between conceptual models and mental models is missing: students don't know how to get from one to the other, and they don't get taught how to do it.

Moreover, students may not actually recognise conceptual models as such. We should explicitly tell students whenever we are presenting a conceptual model.

We should explicitly teach students how to construct good mental models from conceptual models and otherwise, via modelling.

We should explicitly model the processes of developing conceptual models from the concrete, and developing mental models from the conceptual and/or the concrete.

Fixing misconceptions

Students don't arrive as blank slates.

They have pre-existing mental frameworks and models.

It is very easy to make minor adjustments to a model, so it is relatively easy for good students to extend their knowledge and understanding.

It is very difficult to replace an incorrect model with a correct one (regardless of whether or not the student wants to). This is why misconceptions (such as errors in basic algebra) can be so hard to fix.

To fix misconceptions:

- Doing lots of example problems doesn't work. (This just entrenches either the wrong model, or the idea that the student can't do maths.)
- Telling someone what they should think doesn't work. (Consider how few debates on politics or religion actually change someone's deeply held beliefs, regardless of the quality of the arguments.)
- Not only do these two approaches not work to correct wrong mental models, but they actually cause damage. If a student has a mental model for something, then they believe they understand it. If they get all the questions wrong, or get told they are doing it wrong, they can begin to doubt their mathematical reasoning skills in general, leading to maths anxiety and resulting in the student hating, fearing and being frustrated by maths.

It seems that really the only way to effect a model change is to:

1. get the person to predict something with their model …
2. and **discover for themself** that it is completely wrong, …
3. then show them the correct model, and allow them to demonstrate that it solves the problem.

Part 3 What to do?

Here are a few techniques recommended in the research literature for developing good mental models. Many, but not all, are summarised from earlier parts of this paper.

- Allow students to work with concrete models and explicitly teach them how to form mental models from these.
- Explicitly tell students whenever you present a conceptual model.
- Explicitly teach students how to develop a conceptual model from a concrete model.
- Explicitly teach students how to develop a mental model from a conceptual model.
- Allow enough time for students to develop mental models. If a curriculum is too rushed, students will resort to memorizing procedures.
- Use analogies (and/or narrative) to tap into existing, correct mental models to assist in correct formation of new mental models (for example, electricity flow can be taught as analogous to water flow through pipes).
- Actively and purposefully employing imaginative skills may enhance mental modelling skills.
- Design questions that force students to confront and address common contradictions and misconceptions.
- Have students write conceptual explanations of topics, or verbally explain to peers. Require them to use picture language (e.g. diagrams, illustrations, analogies) and not simply list the steps in a procedure.

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