Representation theory of the monoid of all partial functions on a set and other Ehresmann semigroups

Itamar Stein

Shamoon College of Engineering

Abend seminar
October 15, 2020

Joint work with Stuart Margolis
Bar-Ilan University

Semigroup algebras

- S - finite semigroup.
- $\mathbb{k}$ - field.
- $\mathbb{k}S$ - semigroup algebra.

$$\mathbb{k}S = \{\sum k_i s_i \mid k_i \in \mathbb{k} \quad s_i \in S\}$$

$\mathbb{k}S$ is an associative, and finite dimensional $\mathbb{k}$-algebra.
In this talk it will be also unital.

- For this talk $\mathbb{k} = \mathbb{C}$.
- $\mathbb{C}S$ is usually not semisimple

Semigroup algebras

- S - finite semigroup.
- $\mathbb{k}$ - field.
- $\mathbb{k}S$ - semigroup algebra.

$$\mathbb{k}S = \{\sum k_i s_i \mid k_i \in \mathbb{k} \quad s_i \in S\}$$

$\mathbb{k}S$ is an associative, and finite dimensional $\mathbb{k}$-algebra. In this talk it will be also unital.

- For this talk $\mathbb{k} = \mathbb{C}$.
- $\mathbb{C}S$ is usually not semisimple

Semigroup algebras

- S - finite semigroup.
- $\Bbbk$ - field.
- $\Bbbk S$ - semigroup algebra.

$$\Bbbk S = \{\sum k_i s_i \mid k_i \in \Bbbk \quad s_i \in S\}$$

$\Bbbk S$ is an associative, and finite dimensional $\Bbbk$-algebra.
In this talk it will be also unital.

- For this talk $\Bbbk = \mathbb{C}$.
- $\mathbb{C}S$ is usually not semisimple

Semigroup algebras

- S - finite semigroup.
- $\mathbb{k}$ - field.
- $\mathbb{k}S$ - semigroup algebra.

$$\mathbb{k}S = \{\sum k_i s_i \mid k_i \in \mathbb{k} \quad s_i \in S\}$$

$\mathbb{k}S$ is an associative, and finite dimensional $\mathbb{k}$-algebra.
In this talk it will be also unital.

- For this talk $\mathbb{k} = \mathbb{C}$.
- $\mathbb{C}S$ is usually not semisimple

Semigroup algebras

- S - finite semigroup.
- $\Bbbk$ - field.
- $\Bbbk S$ - semigroup algebra.

$$\Bbbk S = \{\sum k_i s_i \mid k_i \in \Bbbk \quad s_i \in S\}$$

  $\Bbbk S$ is an associative, and finite dimensional $\Bbbk$-algebra.
  In this talk it will be also unital.
- For this talk $\Bbbk = \mathbb{C}$.
- $\mathbb{C}S$ is usually not semisimple

Semigroup algebras

- S - finite semigroup.
- $\mathbb{k}$ - field.
- $\mathbb{k}S$ - semigroup algebra.

$$\mathbb{k}S = \{\sum k_i s_i \mid k_i \in \mathbb{k} \quad s_i \in S\}$$

  $\mathbb{k}S$ is an associative, and finite dimensional $\mathbb{k}$-algebra.
  In this talk it will be also unital.
- For this talk $\mathbb{k} = \mathbb{C}$.
- $\mathbb{C}S$ is usually not semisimple

Semigroup algebras

- S - finite semigroup.
- $\Bbbk$ - field.
- $\Bbbk S$ - semigroup algebra.

$$\Bbbk S = \{\sum k_i s_i \mid k_i \in \Bbbk \quad s_i \in S\}$$

  $\Bbbk S$ is an associative, and finite dimensional $\Bbbk$-algebra.
  In this talk it will be also unital.
- For this talk $\Bbbk = \mathbb{C}$.
- $\mathbb{C}S$ is usually not semisimple!

Simple & Projective modules

- A - associative, unital, finite dimensional $\Bbbk$-algebra.
- **Simple module** - a module with no proper submodules.
  - Finite up to isomorphism.
- **Ind. module** - cannot be written as direct sum of non-zero modules ($P = P_1 \oplus P_2$)
- **Projective module** - P is projective if $\text{Hom}(P, -)$ is an exact functor.
- **Equivalently:** P is a direct summand of A^k.
- **Ind. Projective** - $P \simeq Ae$ where e is a primitive idempotent.
  - Finite up to isomorphism.

Simple & Projective modules

- A - associative, unital, finite dimensional $\mathbb{k}$-algebra.
- **Simple module** - a module with no proper submodules.
  - Finite up to isomorphism.
- **Ind. module** - cannot be written as direct sum of non-zero modules ($P = P_1 \oplus P_2$)
- **Projective module** - P is projective if $\mathrm{Hom}(P, -)$ is an exact functor.
- **Equivalently:** P is a direct summand of A^k.
- **Ind. Projective** - $P \simeq Ae$ where e is a primitive idempotent.
  - Finite up to isomorphism.

Simple & Projective modules

- A - associative, unital, finite dimensional $\mathbb{k}$-algebra.
- **Simple module** - a module with no proper submodules.
  - Finite up to isomorphism.
- **Ind. module** - cannot be written as direct sum of non-zero modules ($P = P_1 \oplus P_2$)
- **Projective module** - P is projective if $\mathrm{Hom}(P, -)$ is an exact functor.
- **Equivalently:** P is a direct summand of A^k.
- **Ind. Projective** - $P \simeq Ae$ where e is a primitive idempotent.
  - Finite up to isomorphism.

Simple & Projective modules

- A - associative, unital, finite dimensional $\mathbb{k}$-algebra.
- **Simple module** - a module with no proper submodules.
  - Finite up to isomorphism.
- **Ind. module** - cannot be written as direct sum of non-zero modules $(P = P_1 \oplus P_2)$
- **Projective module** - P is projective if $\text{Hom}(P, -)$ is an exact functor.
- **Equivalently:** P is a direct summand of A^k.
- **Ind. Projective** - $P \simeq Ae$ where e is a primitive idempotent.
  - Finite up to isomorphism.

Simple & Projective modules

- A - associative, unital, finite dimensional $\mathbb{k}$-algebra.
- **Simple module** - a module with no proper submodules.
  - Finite up to isomorphism.
- **Ind. module** - cannot be written as direct sum of non-zero modules $(P = P_1 \oplus P_2)$
- **Projective module** - P is projective if $\text{Hom}(P, -)$ is an exact functor.
- **Equivalently:** P is a direct summand of A^k.
- **Ind. Projective** - $P \simeq Ae$ where e is a primitive idempotent.
  - Finite up to isomorphism.

Simple & Projective modules

- A - associative, unital, finite dimensional $\mathbb{k}$-algebra.
- **Simple module** - a module with no proper submodules.
  - Finite up to isomorphism.
- **Ind. module** - cannot be written as direct sum of non-zero modules ($P = P_1 \oplus P_2$)
- **Projective module** - P is projective if $\mathrm{Hom}(P, -)$ is an exact functor.
- **Equivalently:** P is a direct summand of A^k.
- **Ind. Projective** - $P \simeq Ae$ where e is a primitive idempotent.
  - Finite up to isomorphism.

Simple & Projective modules

- A - associative, unital, finite dimensional $\mathbb{k}$-algebra.
- **Simple module** - a module with no proper submodules.
  - Finite up to isomorphism.
- **Ind. module** - cannot be written as direct sum of non-zero modules ($P = P_1 \oplus P_2$)
- **Projective module** - P is projective if $\operatorname{Hom}(P, -)$ is an exact functor.
- **Equivalently:** P is a direct summand of A^k.
- **Ind. Projective** - $P \simeq Ae$ where e is a primitive idempotent.
  - Finite up to isomorphism.

Simple & Projective modules

- A - associative, unital, finite dimensional $\mathbb{k}$-algebra.
- **Simple module** - a module with no proper submodules.
  - » Finite up to isomorphism.
- **Ind. module** - cannot be written as direct sum of non-zero modules ($P = P_1 \oplus P_2$)
- **Projective module** - P is projective if $\text{Hom}(P, -)$ is an exact functor.
- **Equivalently:** P is a direct summand of A^k.
- **Ind. Projective** - $P \simeq Ae$ where e is a primitive idempotent.
  - » Finite up to isomorphism.

Goal\Outline

- Description of simple and Ind. Projective modules of $\mathbb{C}S$ in the following cases:

1. $S = \mathcal{PT}_n$. The monoid of partial functions on $\{1, \ldots, n\}$.
2. Generalization: S is a certain kind of Ehresmann semigroup.

Goal\Outline

- Description of simple and Ind. Projective modules of $\mathbb{C}S$ in the following cases:

  1. $S = \mathcal{PT}_n$. The monoid of partial functions on $\{1, \ldots, n\}$.
  2. Generalization: S is a certain kind of Ehresmann semigroup.

Goal\Outline

- Description of simple and Ind. Projective modules of $\mathbb{C}S$ in the following cases:

1. $S = \mathcal{PT}_n$. The monoid of partial functions on $\{1, \dots, n\}$.
2. Generalization: S is a certain kind of Ehresmann semigroup.

Goal\Outline

- Description of simple and Ind. Projective modules of $\mathbb{C}S$ in the following cases:

  1. $S = \mathcal{PT}_n$. The monoid of partial functions on $\{1, \ldots, n\}$.

  2. Generalization: S is a certain kind of Ehresmann semigroup.

Partial functions

Example

$f \in \mathcal{PT}_4$ given by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & \varnothing & 2 & 3 \end{pmatrix}$$

$(f(1) = f(3) = 2, \quad f(4) = 3, \quad f(2)$ undefined$)$

$$\mathrm{dom}(f) = \{1,3,4\}, \quad \mathrm{im}(f) = \{2,3\}, \quad \mathrm{rank}(f) = |\,\mathrm{im}(f)| = 2$$

Definition

The kernel of f is an equivalence relation on $\mathrm{dom}(f)$ defined by $a \sim b \iff f(a) = f(b)$.

In the above example the kernel classes are $\{1,3\}, \{4\}$

Partial functions

Example

$f \in \mathcal{PT}_4$ given by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & \varnothing & 2 & 3 \end{pmatrix}$$

$(f(1) = f(3) = 2, \quad f(4) = 3, \quad f(2)$ undefined$)$

$$\mathrm{dom}(f) = \{1,3,4\}, \quad \mathrm{im}(f) = \{2,3\}, \quad \mathrm{rank}(f) = |\,\mathrm{im}(f)| = 2$$

Definition

The kernel of f is an equivalence relation on $\mathrm{dom}(f)$ defined by $a \sim b \iff f(a) = f(b)$.

In the above example the kernel classes are $\{1,3\}, \{4\}$

Partial functions

Example

$f \in \mathcal{PT}_4$ given by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & \varnothing & 2 & 3 \end{pmatrix}$$

$(f(1) = f(3) = 2, \quad f(4) = 3, \quad f(2)$ undefined$)$

$$\mathrm{dom}(f) = \{1,3,4\}, \quad \mathrm{im}(f) = \{2,3\}, \quad \mathrm{rank}(f) = |\,\mathrm{im}(f)| = 2$$

Definition

The kernel of f is an equivalence relation on $\mathrm{dom}(f)$ defined by $a \sim b \iff f(a) = f(b)$.

In the above example the kernel classes are $\{1,3\}, \{4\}$

Partial functions

Example

$f \in \mathcal{PT}_4$ given by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & \varnothing & 2 & 3 \end{pmatrix}$$

$(f(1) = f(3) = 2, \quad f(4) = 3, \quad f(2)$ undefined$)$

$$\mathrm{dom}(f) = \{1,3,4\}, \quad \mathrm{im}(f) = \{2,3\}, \quad \mathrm{rank}(f) = |\,\mathrm{im}(f)| = 2$$

Definition

The kernel of f is an equivalence relation on $\mathrm{dom}(f)$ defined by
$a \sim b \iff f(a) = f(b)$.

In the above example the kernel classes are $\{1,3\}, \{4\}$

Construction of simple modules of $\mathcal{PT}_n$

$\mathcal{L}$-classes

- Partial functions $f, g \in \mathcal{PT}_n$ are $\mathcal{L}$-related $\iff$ $\text{dom}(f) = \text{dom}(g)$ and $\ker(f) = \ker(g)$.
- $f \mathcal{L} g \implies \text{rank}(f) = \text{rank}(g)$. So for $\mathcal{L}$-class L we can define $\text{rank}(L)$.
- Every $\mathcal{L}$ class contains an idempotent $e \in L$.
- Let L be an $\mathcal{L}$-class of rank k and let $e \in L$. G_e - The maximal subgroup at e (with unit element e) is isomorphic to S_k (the symmetric group of degree k).

$\mathcal{L}$-classes

- Partial functions $f, g \in \mathcal{PT}_n$ are $\mathcal{L}$-related $\iff$ $\text{dom}(f) = \text{dom}(g)$ and $\ker(f) = \ker(g)$.
- $f \mathcal{L} g \implies \text{rank}(f) = \text{rank}(g)$. So for $\mathcal{L}$-class L we can define $\text{rank}(L)$.
- Every $\mathcal{L}$ class contains an idempotent $e \in L$.
- Let L be an $\mathcal{L}$-class of rank k and let $e \in L$. G_e - The maximal subgroup at e (with unit element e) is isomorphic to S_k (the symmetric group of degree k).

$\mathcal{L}$-classes

- Partial functions $f, g \in \mathcal{PT}_n$ are $\mathcal{L}$-related $\iff$ $\mathrm{dom}(f) = \mathrm{dom}(g)$ and $\ker(f) = \ker(g)$.
- $f\mathcal{L}g \implies \mathrm{rank}(f) = \mathrm{rank}(g)$. So for $\mathcal{L}$-class L we can define $\mathrm{rank}(L)$.
- Every $\mathcal{L}$ class contains an idempotent $e \in L$.
- Let L be an $\mathcal{L}$-class of rank k and let $e \in L$. G_e - The maximal subgroup at e (with unit element e) is isomorphic to S_k (the symmetric group of degree k).

$\mathcal{L}$-classes

- Partial functions $f, g \in \mathcal{PT}_n$ are $\mathcal{L}$-related $\iff$ $\text{dom}(f) = \text{dom}(g)$ and $\ker(f) = \ker(g)$.
- $f \mathcal{L} g \implies \text{rank}(f) = \text{rank}(g)$. So for $\mathcal{L}$-class L we can define $\text{rank}(L)$.
- Every $\mathcal{L}$ class contains an idempotent $e \in L$.
- Let L be an $\mathcal{L}$-class of rank k and let $e \in L$. G_e - The maximal subgroup at e (with unit element e) is isomorphic to S_k (the symmetric group of degree k).

$\mathcal{L}$-classes

- Partial functions $f, g \in \mathcal{PT}_n$ are $\mathcal{L}$-related $\iff$ $\mathrm{dom}(f) = \mathrm{dom}(g)$ and $\ker(f) = \ker(g)$.
- $f \mathcal{L} g \implies \mathrm{rank}(f) = \mathrm{rank}(g)$. So for $\mathcal{L}$-class L we can define $\mathrm{rank}(L)$.
- Every $\mathcal{L}$ class contains an idempotent $e \in L$.
- Let L be an $\mathcal{L}$-class of rank k and let $e \in L$. G_e - The maximal subgroup at e (with unit element e) is isomorphic to S_k (the symmetric group of degree k).

$\mathcal{L}$-classes

- Partial functions $f, g \in \mathcal{PT}_n$ are $\mathcal{L}$-related $\iff$ $\text{dom}(f) = \text{dom}(g)$ and $\ker(f) = \ker(g)$.
- $f \mathcal{L} g \implies \text{rank}(f) = \text{rank}(g)$. So for $\mathcal{L}$-class L we can define $\text{rank}(L)$.
- Every $\mathcal{L}$ class contains an idempotent $e \in L$.
- Let L be an $\mathcal{L}$-class of rank k and let $e \in L$. G_e - The maximal subgroup at e (with unit element e) is isomorphic to S_k (the symmetric group of degree k).

Natural actions on an $\mathcal{L}$-class

Let L be an $\mathcal{L}$-class of rank k and let $e \in L$ be an idempotent. Then $G_e \simeq S_k$ acts on the right of L by right multiplication

$$f * g = fg \quad (f \in L, \quad g \in G_e)$$

Natural actions on an $\mathcal{L}$-class

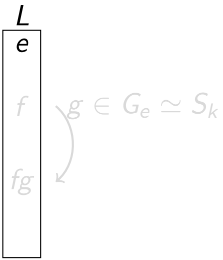
Let L be an $\mathcal{L}$-class of rank k and let $e \in L$ be an idempotent. Then $G_e \simeq S_k$ acts on the right of L by right multiplication

$$f * g = fg \quad (f \in L, \quad g \in G_e)$$

Natural actions on an $\mathcal{L}$-class

Let L be an $\mathcal{L}$-class of rank k and let $e \in L$ be an idempotent. Then $G_e \simeq S_k$ acts on the right of L by right multiplication

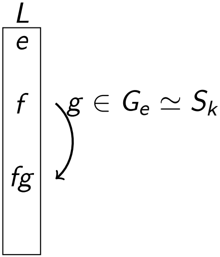
$$f * g = fg \quad (f \in L, \quad g \in G_e)$$



Natural actions on an $\mathcal{L}$-class

Let L be an $\mathcal{L}$-class of rank k and let $e \in L$ be an idempotent. Then $G_e \simeq S_k$ acts on the right of L by right multiplication

$$f * g = fg \quad (f \in L, \quad g \in G_e)$$



Natural actions on an $\mathcal{L}$-class

Let L be an $\mathcal{L}$-class of rank k and let $e \in L$ be an idempotent. Then $\mathcal{PT}_n$ ***partially*** acts on the left of L

$$h * f = \begin{cases} hf & hf \in L \\ \text{undefined} & hf \notin L \end{cases} \quad (f \in L, \quad h \in \mathcal{PT}_n)$$

Natural actions on an $\mathcal{L}$-class

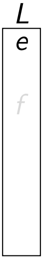
Let L be an $\mathcal{L}$-class of rank k and let $e \in L$ be an idempotent. Then $\mathcal{PT}_n$ ***partially*** acts on the left of L

$$h * f = \begin{cases} hf & hf \in L \\ \text{undefined} & hf \notin L \end{cases} \quad (f \in L, \quad h \in \mathcal{PT}_n)$$

Natural actions on an $\mathcal{L}$-class

Let L be an $\mathcal{L}$-class of rank k and let $e \in L$ be an idempotent. Then $\mathcal{PT}_n$ ***partially*** acts on the left of L

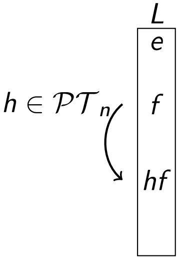
$$h * f = \begin{cases} hf & hf \in L \\ \text{undefined} & hf \notin L \end{cases} \quad (f \in L, \quad h \in \mathcal{PT}_n)$$



Natural actions on an $\mathcal{L}$-class

Let L be an $\mathcal{L}$-class of rank k and let $e \in L$ be an idempotent. Then $\mathcal{PT}_n$ ***partially*** acts on the left of L

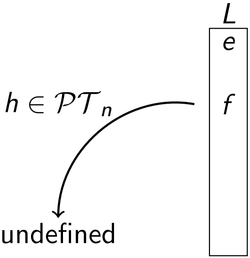
$$h * f = \begin{cases} hf & hf \in L \\ \text{undefined} & hf \notin L \end{cases} \quad (f \in L, \quad h \in \mathcal{PT}_n)$$



Natural actions on an $\mathcal{L}$-class

Let L be an $\mathcal{L}$-class of rank k and let $e \in L$ be an idempotent. Then $\mathcal{PT}_n$ ***partially*** acts on the left of L

$$h * f = \begin{cases} hf & hf \in L \\ \text{undefined} & hf \notin L \end{cases} \quad (f \in L, \quad h \in \mathcal{PT}_n)$$



Induced left Schützenberger module

- This induces a structure of $\mathbb{C}\mathcal{PT}_n - \mathbb{C}S_k$ bi-module on $\mathbb{C}L = \{\sum \alpha_i f_i \mid \alpha_i \in \mathbb{C}, \quad f_i \in L\}$ defined on basis elements by

$$f * g = fg \quad (f \in L, \quad g \in G_e)$$

$$h * f = \begin{cases} hf & hf \in L \\ 0 & hf \notin L \end{cases} \quad (f \in L, \quad h \in \mathcal{PT}_n)$$

- Let S^λ be a simple module of $\mathbb{C}S_k$. So $\mathbb{C}L \otimes S^\lambda$ is a $\mathbb{C}\mathcal{PT}_n$-module.

Theorem (Munn, Ponizovskii (1950∼), Margolis & Steinberg (2011)-modern proof)

$\mathbb{C}L \otimes S^\lambda$ is a simple $\mathbb{C}\mathcal{PT}_n$-module and every simple module is of this form (for some choice of $\mathcal{L}$-class L and $\lambda \vdash k$).

Induced left Schützenberger module

- This induces a structure of $\mathbb{C}\mathcal{PT}_n - \mathbb{C}S_k$ bi-module on $\mathbb{C}L = \{\sum \alpha_i f_i \mid \alpha_i \in \mathbb{C}, \quad f_i \in L\}$ defined on basis elements by

$$f * g = fg \quad (f \in L, \quad g \in G_e)$$

$$h * f = \begin{cases} hf & hf \in L \\ 0 & hf \notin L \end{cases} \quad (f \in L, \quad h \in \mathcal{PT}_n)$$

- Let S^λ be a simple module of $\mathbb{C}S_k$. So $\mathbb{C}L \otimes S^\lambda$ is a $\mathbb{C}\mathcal{PT}_n$-module.

Theorem (Munn, Ponizovskii (1950∼), Margolis & Steinberg (2011)-modern proof)

$\mathbb{C}L \otimes S^\lambda$ is a simple $\mathbb{C}\mathcal{PT}_n$-module and every simple module is of this form (for some choice of $\mathcal{L}$-class L and $\lambda \vdash k$).

Induced left Schützenberger module

- This induces a structure of $\mathbb{C}\mathcal{PT}_n - \mathbb{C}S_k$ bi-module on $\mathbb{C}L = \{\sum \alpha_i f_i \mid \alpha_i \in \mathbb{C}, \quad f_i \in L\}$ defined on basis elements by

$$f * g = fg \quad (f \in L, \quad g \in G_e)$$

$$h * f = \begin{cases} hf & hf \in L \\ 0 & hf \notin L \end{cases} \quad (f \in L, \quad h \in \mathcal{PT}_n)$$

- Let S^λ be a simple module of $\mathbb{C}S_k$. So $\mathbb{C}L \otimes S^\lambda$ is a $\mathbb{C}\mathcal{PT}_n$-module.

Theorem (Munn, Ponizovskii (1950∼), Margolis & Steinberg (2011)-modern proof)

$\mathbb{C}L \otimes S^\lambda$ is a simple $\mathbb{C}\mathcal{PT}_n$-module and every simple module is of this form (for some choice of $\mathcal{L}$-class L and $\lambda \vdash k$).

Induced left Schützenberger module

- This induces a structure of $\mathbb{C}\mathcal{PT}_n - \mathbb{C}S_k$ bi-module on $\mathbb{C}L = \{\sum \alpha_i f_i \mid \alpha_i \in \mathbb{C}, \quad f_i \in L\}$ defined on basis elements by

$$f * g = fg \quad (f \in L, \quad g \in G_e)$$

$$h * f = \begin{cases} hf & hf \in L \\ 0 & hf \notin L \end{cases} \quad (f \in L, \quad h \in \mathcal{PT}_n)$$

- Let S^λ be a simple module of $\mathbb{C}S_k$. So $\mathbb{C}L \otimes S^\lambda$ is a $\mathbb{C}\mathcal{PT}_n$-module.

Theorem (Munn, Ponizovskii (1950∼), Margolis & Steinberg (2011)-modern proof)

$\mathbb{C}L \otimes S^\lambda$ is a simple $\mathbb{C}\mathcal{PT}_n$-module and every simple module is of this form (for some choice of $\mathcal{L}$-class L and $\lambda \vdash k$).

Isomorphic copies

Remark

$$\mathbb{C}L_1 \otimes S^{\lambda_1} \simeq \mathbb{C}L_2 \otimes S^{\lambda_2} \iff \mathrm{rank}(L_1) = \mathrm{rank}(L_2), \quad \lambda_1 = \lambda_2$$

Construction of ind. projective modules of $\mathcal{PT}_n$

$\widetilde{\mathcal{L}}_E$-classes

- Partial functions $f, g \in \mathcal{PT}_n$ are $\widetilde{\mathcal{L}}_E$-related $\iff$ dom$(f) =$ dom(g). Clearly $\mathcal{L} \subseteq \widetilde{\mathcal{L}}_E$.

- For every $A \subseteq \{1, \ldots, n\}$ denote by id_A the partial identity of A.

$$\mathrm{id}_A(i) = \begin{cases} i & i \in A \\ \text{undefined} & i \notin A \end{cases}$$

- Note that every $\widetilde{\mathcal{L}}_E$ class contains one partial identity.

$\widetilde{\mathcal{L}}_E$-classes

- Partial functions $f, g \in \mathcal{PT}_n$ are $\widetilde{\mathcal{L}}_E$-related $\iff$ dom$(f) =$ dom(g). Clearly $\mathcal{L} \subseteq \widetilde{\mathcal{L}}_E$.

- For every $A \subseteq \{1, \ldots, n\}$ denote by id_A the partial identity of A.

$$\mathrm{id}_A(i) = \begin{cases} i & i \in A \\ \text{undefined} & i \notin A \end{cases}$$

- Note that every $\widetilde{\mathcal{L}}_E$ class contains one partial identity.

$\widetilde{\mathcal{L}}_E$-classes

- Partial functions $f, g \in \mathcal{PT}_n$ are $\widetilde{\mathcal{L}}_E$-related $\iff$ dom$(f) =$ dom(g). Clearly $\mathcal{L} \subseteq \widetilde{\mathcal{L}}_E$.
- For every $A \subseteq \{1, \ldots, n\}$ denote by id_A the partial identity of A.

$$\mathrm{id}_A(i) = \begin{cases} i & i \in A \\ \text{undefined} & i \notin A \end{cases}$$

- Note that every $\widetilde{\mathcal{L}}_E$ class contains one partial identity.

$\widetilde{\mathcal{L}}_E$-classes

- Partial functions $f, g \in \mathcal{PT}_n$ are $\widetilde{\mathcal{L}}_E$-related $\iff$ $\text{dom}(f) = \text{dom}(g)$. Clearly $\mathcal{L} \subseteq \widetilde{\mathcal{L}}_E$.
- For every $A \subseteq \{1, \ldots, n\}$ denote by id_A the partial identity of A.

$$\text{id}_A(i) = \begin{cases} i & i \in A \\ \text{undefined} & i \notin A \end{cases}$$

- Note that every $\widetilde{\mathcal{L}}_E$ class contains one partial identity.

$\widetilde{\mathcal{L}}_E$-classes

- Partial functions $f, g \in \mathcal{PT}_n$ are $\widetilde{\mathcal{L}}_E$-related $\iff$ dom$(f) =$ dom(g). Clearly $\mathcal{L} \subseteq \widetilde{\mathcal{L}}_E$.

- For every $A \subseteq \{1, \dots, n\}$ denote by id_A the partial identity of A.

$$\mathrm{id}_A(i) = \begin{cases} i & i \in A \\ \text{undefined} & i \notin A \end{cases}$$

- Note that every $\widetilde{\mathcal{L}}_E$ class contains one partial identity.

Natural actions on an $\widetilde{\mathcal{L}}_E$-class

Let $\widetilde{L}$ be an $\widetilde{\mathcal{L}}_E$-class and let $\mathrm{id}_A \in \tilde{L}$. Then S_A acts on the right of $\widetilde{L}$ by right multiplication

$$f * g = fg \quad (f \in \widetilde{L}, \quad g \in S_A)$$

Natural actions on an $\widetilde{\mathcal{L}}_E$-class

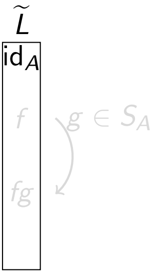
Let $\widetilde{L}$ be an $\widetilde{\mathcal{L}}_E$-class and let $\mathrm{id}_A \in \tilde{L}$. Then S_A acts on the right of $\widetilde{L}$ by right multiplication

$$f * g = fg \quad (f \in \widetilde{L}, \quad g \in S_A)$$

Natural actions on an $\widetilde{\mathcal{L}}_E$-class

Let $\widetilde{L}$ be an $\widetilde{\mathcal{L}}_E$-class and let $\mathrm{id}_A \in \tilde{L}$. Then S_A acts on the right of $\widetilde{L}$ by right multiplication

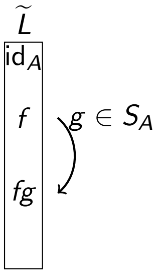
$$f * g = fg \quad (f \in \widetilde{L}, \quad g \in S_A)$$



Natural actions on an $\widetilde{\mathcal{L}}_E$-class

Let $\widetilde{L}$ be an $\widetilde{\mathcal{L}}_E$-class and let $\mathrm{id}_A \in \tilde{L}$. Then S_A acts on the right of $\widetilde{L}$ by right multiplication

$$f * g = fg \quad (f \in \widetilde{L}, \quad g \in S_A)$$



Natural actions on an $\widetilde{\mathcal{L}}_E$-class

Let $\widetilde{L}$ be an $\widetilde{\mathcal{L}}_E$-class. Then $\mathcal{PT}_n$ ***partially*** acts on the left of $\widetilde{L}$

$$h * f = \begin{cases} hf & hf \in \widetilde{L} \\ \text{undefined} & hf \notin \widetilde{L} \end{cases} \quad (f \in \widetilde{L}, \quad h \in \mathcal{PT}_n)$$

Natural actions on an $\widetilde{\mathcal{L}}_E$-class

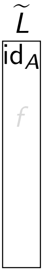
Let $\widetilde{L}$ be an $\widetilde{\mathcal{L}}_E$-class. Then $\mathcal{PT}_n$ ***partially*** acts on the left of $\widetilde{L}$

$$h * f = \begin{cases} hf & hf \in \widetilde{L} \\ \text{undefined} & hf \notin \widetilde{L} \end{cases} \quad (f \in \widetilde{L}, \quad h \in \mathcal{PT}_n)$$

Natural actions on an $\widetilde{\mathcal{L}}_E$-class

Let $\widetilde{L}$ be an $\widetilde{\mathcal{L}}_E$-class. Then $\mathcal{PT}_n$ ***partially*** acts on the left of $\widetilde{L}$

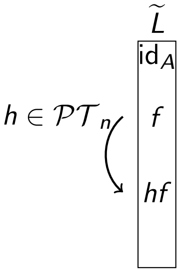
$$h * f = \begin{cases} hf & hf \in \widetilde{L} \\ \text{undefined} & hf \notin \widetilde{L} \end{cases} \quad (f \in \widetilde{L}, \quad h \in \mathcal{PT}_n)$$



Natural actions on an $\widetilde{\mathcal{L}}_E$-class

Let $\widetilde{L}$ be an $\widetilde{\mathcal{L}}_E$-class. Then $\mathcal{PT}_n$ ***partially*** acts on the left of $\widetilde{L}$

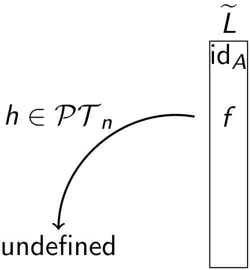
$$h * f = \begin{cases} hf & hf \in \widetilde{L} \\ \text{undefined} & hf \notin \widetilde{L} \end{cases} \quad (f \in \widetilde{L}, \quad h \in \mathcal{PT}_n)$$



Natural actions on an $\widetilde{\mathcal{L}}_E$-class

Let $\widetilde{L}$ be an $\widetilde{\mathcal{L}}_E$-class. Then $\mathcal{PT}_n$ ***partially*** acts on the left of $\widetilde{L}$

$$h * f = \begin{cases} hf & hf \in \widetilde{L} \\ \text{undefined} & hf \notin \widetilde{L} \end{cases} \quad (f \in \widetilde{L}, \quad h \in \mathcal{PT}_n)$$



Ind. projective modules

- This induces a structure of $\mathbb{C}\mathcal{PT}_n - \mathbb{C}S_A$ bi-module on $\mathbb{C}\widetilde{L} = \{\sum \alpha_i f_i \mid \alpha_i \in \mathbb{C}, \quad f_i \in \widetilde{L}\}$ as before.
- Let S^λ be a simple module of $\mathbb{C}S_A$. So $\mathbb{C}\widetilde{L} \otimes S^\lambda$ is a $\mathbb{C}\mathcal{PT}_n$-module.

Theorem (Margolis & IS)

$\mathbb{C}\widetilde{L} \otimes S^\lambda$ is an ind. projective $\mathbb{C}\mathcal{PT}_n$-module and every ind. projective module is of this form (for some choice of $\widetilde{\mathcal{L}}_E$-class $\widetilde{L}$ and $\lambda \vdash |A|$).

Remark

$$\mathbb{C}\widetilde{L}_1 \otimes S^{\lambda_1} \simeq \mathbb{C}\widetilde{L}_2 \otimes S^{\lambda_2} \iff |\mathrm{dom}(\widetilde{L}_1)| = |\mathrm{dom}(\widetilde{L}_2)|, \quad \lambda_1 = \lambda_2$$

Ind. projective modules

- This induces a structure of $\mathbb{C}\mathcal{PT}_n - \mathbb{C}S_A$ bi-module on $\mathbb{C}\widetilde{L} = \{\sum \alpha_i f_i \mid \alpha_i \in \mathbb{C}, \quad f_i \in \widetilde{L}\}$ as before.
- Let S^λ be a simple module of $\mathbb{C}S_A$. So $\mathbb{C}\widetilde{L} \otimes S^\lambda$ is a $\mathbb{C}\mathcal{PT}_n$-module.

Theorem (Margolis & IS)

$\mathbb{C}\widetilde{L} \otimes S^\lambda$ is an ind. projective $\mathbb{C}\mathcal{PT}_n$-module and every ind. projective module is of this form (for some choice of $\widetilde{\mathcal{L}}_E$-class $\widetilde{L}$ and $\lambda \vdash |A|$).

Remark

$$\mathbb{C}\widetilde{L_1} \otimes S^{\lambda_1} \simeq \mathbb{C}\widetilde{L_2} \otimes S^{\lambda_2} \iff |\text{dom}(\widetilde{L_1})| = |\text{dom}(\widetilde{L_2})|, \quad \lambda_1 = \lambda_2$$

Ind. projective modules

- This induces a structure of $\mathbb{C}\mathcal{PT}_n - \mathbb{C}S_A$ bi-module on $\mathbb{C}\widetilde{L} = \{\sum \alpha_i f_i \mid \alpha_i \in \mathbb{C}, \quad f_i \in \widetilde{L}\}$ as before.
- Let S^λ be a simple module of $\mathbb{C}S_A$. So $\mathbb{C}\widetilde{L} \otimes S^\lambda$ is a $\mathbb{C}\mathcal{PT}_n$-module.

Theorem (Margolis & IS)

$\mathbb{C}\widetilde{L} \otimes S^\lambda$ is an ind. projective $\mathbb{C}\mathcal{PT}_n$-module and every ind. projective module is of this form (for some choice of $\widetilde{\mathcal{L}}_E$-class $\widetilde{L}$ and $\lambda \vdash |A|$).

Remark

$$\mathbb{C}\widetilde{L}_1 \otimes S^{\lambda_1} \simeq \mathbb{C}\widetilde{L}_2 \otimes S^{\lambda_2} \iff |\mathrm{dom}(\widetilde{L}_1)| = |\mathrm{dom}(\widetilde{L}_2)|, \quad \lambda_1 = \lambda_2$$

Ind. projective modules

- This induces a structure of $\mathbb{C}\mathcal{PT}_n - \mathbb{C}S_A$ bi-module on $\mathbb{C}\widetilde{L} = \{\sum \alpha_i f_i \mid \alpha_i \in \mathbb{C}, \quad f_i \in \widetilde{L}\}$ as before.
- Let S^λ be a simple module of $\mathbb{C}S_A$. So $\mathbb{C}\widetilde{L} \otimes S^\lambda$ is a $\mathbb{C}\mathcal{PT}_n$-module.

Theorem (Margolis & IS)

$\mathbb{C}\widetilde{L} \otimes S^\lambda$ is an ind. projective $\mathbb{C}\mathcal{PT}_n$-module and every ind. projective module is of this form (for some choice of $\widetilde{\mathcal{L}}_E$-class $\widetilde{L}$ and $\lambda \vdash |A|$).

Remark

$$\mathbb{C}\widetilde{L}_1 \otimes S^{\lambda_1} \simeq \mathbb{C}\widetilde{L}_2 \otimes S^{\lambda_2} \iff |\mathrm{dom}(\widetilde{L}_1)| = |\mathrm{dom}(\widetilde{L}_2)|, \quad \lambda_1 = \lambda_2$$

Dimension of Ind. Projective modules

If $|\operatorname{dom}(\widetilde{L})| = k$ and $\lambda \vdash k$ then

$$\dim(\mathbb{C}\widetilde{L} \otimes S^{\lambda}) = \sum_{l=1}^{k} \sum_{\substack{\mu \vDash k \\ |\mu| = l}} \binom{n}{l} K_{\lambda\mu}$$

(where $K_{\lambda\mu}$ is the Kostka number of λ and μ- the number of semistandard Young tableaux with shape λ and content μ).

New? - category approach

- Define a category $\mathcal{E}_n$ as follows. Objects: subsets of $\{1,\ldots,n\}$. Morphisms: $\mathcal{E}_n(A,B)=\{f:A\to B \mid f \text{ onto}\}$

Theorem (IS 2016)

There is an isomorphism of algebras

$$\mathbb{C}\mathcal{PT}_n \simeq \mathbb{C}\mathcal{E}_n$$

- $\mathcal{E}_n$ is an EI-category (every endomorphism is an isomorphism) and there is a known way to describe projective modules of EI-categories.

New? - category approach

- Define a category $\mathcal{E}_n$ as follows. Objects: subsets of $\{1, \ldots, n\}$. Morphisms: $\mathcal{E}_n(A, B) = \{f : A \to B \mid f \text{ onto}\}$

Theorem (IS 2016)

There is an isomorphism of algebras

$$\mathbb{C}\mathcal{PT}_n \simeq \mathbb{C}\mathcal{E}_n$$

- $\mathcal{E}_n$ is an EI-category (every endomorphism is an isomorphism) and there is a known way to describe projective modules of EI-categories.

New? - category approach

- Define a category $\mathcal{E}_n$ as follows. Objects: subsets of $\{1,\ldots,n\}$. Morphisms: $\mathcal{E}_n(A,B)=\{f:A\to B \mid f \text{ onto}\}$

Theorem (IS 2016)

There is an isomorphism of algebras

$$\mathbb{C}\mathcal{PT}_n \simeq \mathbb{C}\mathcal{E}_n$$

- $\mathcal{E}_n$ is an EI-category (every endomorphism is an isomorphism) and there is a known way to describe projective modules of EI-categories.

Blank Whiteboard

However...

- Our result is in semigroup theoretic language.
- Similar to other results in the literature (bands, $\mathcal{R}$-trivial monoids etc.).
- Transparent correspondence between simple and its projective cover. Let L and $\widetilde{L}$ be the $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ -classes of some partial identity id_A. Define

$$\psi : \mathbb{C}\widetilde{L} \to \mathbb{C}L$$

$$\psi(m) = \begin{cases} m & m \in L \\ 0 & m \notin L \end{cases}$$

This induces an $\mathbb{C}\mathcal{PT}_n$-module epimorphism

$$\psi \otimes \mathrm{id} : \mathbb{C}\widetilde{L} \bigotimes V \to \mathbb{C}L \bigotimes V.$$

However...

- Our result is in semigroup theoretic language.
- Similar to other results in the literature (bands, $\mathcal{R}$-trivial monoids etc.).
- Transparent correspondence between simple and its projective cover. Let L and $\widetilde{L}$ be the $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ -classes of some partial identity id_A. Define

$$\psi : \mathbb{C}\widetilde{L} \to \mathbb{C}L$$

$$\psi(m) = \begin{cases} m & m \in L \\ 0 & m \notin L \end{cases}$$

This induces an $\mathbb{C}\mathcal{PT}_n$-module epimorphism

$$\psi \otimes \mathrm{id} : \mathbb{C}\widetilde{L} \bigotimes V \to \mathbb{C}L \bigotimes V.$$

However...

- Our result is in semigroup theoretic language.

- Similar to other results in the literature (bands, $\mathcal{R}$-trivial monoids etc.).

- Transparent correspondence between simple and its projective cover. Let L and $\widetilde{L}$ be the $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ -classes of some partial identity id_A. Define

$$\psi : \mathbb{C}\widetilde{L} \to \mathbb{C}L$$

$$\psi(m) = \begin{cases} m & m \in L \\ 0 & m \notin L \end{cases}$$

This induces an $\mathbb{C}\mathcal{PT}_n$-module epimorphism

$$\psi \otimes \mathrm{id} : \mathbb{C}\widetilde{L} \bigotimes V \to \mathbb{C}L \bigotimes V.$$

General case

General case

Everything said can be generalized to a class of **finite** semigroups satisfying three properties:

1. E-Ehresmann
2. right restriction
3. the corresponding Ehresmann category is EI (every endomorphism is an isomorphism)

General case

Everything said can be generalized to a class of **finite** semigroups satisfying three properties:

1. E-Ehresmann
2. right restriction
3. the corresponding Ehresmann category is EI (every endomorphism is an isomorphism)

General case

Everything said can be generalized to a class of **finite** semigroups satisfying three properties:

1. E-Ehresmann
2. right restriction
3. the corresponding Ehresmann category is EI (every endomorphism is an isomorphism)

General case

Everything said can be generalized to a class of **finite** semigroups satisfying three properties:

1. E-Ehresmann
2. right restriction
3. the corresponding Ehresmann category is EI (every endomorphism is an isomorphism)

General case

Everything said can be generalized to a class of **finite** semigroups satisfying three properties:

1. E-Ehresmann
2. right restriction
3. the corresponding Ehresmann category is EI (every endomorphism is an isomorphism)

General case

Everything said can be generalized to a class of **finite** semigroups satisfying three properties:

1. E-Ehresmann
2. right restriction
3. the corresponding Ehresmann category is EI (every endomorphism is an isomorphism)

Ehresmann semigroups

Definition (Lawson, 1991)

Let S be a semigroup and $E \subseteq S$ be a subsemilattice. S is called E-Ehresmann if the following holds:

- For every $a \in S$ there is a minimal $e \in E$ such that $ea = a$ (denoted a^+).
- For every $a \in S$ there is a minimal $e \in E$ such that $ae = a$ (denoted a^*).
- The following identities hold: $(ab)^+ = (ab^+)^+$ and $(ab)^* = (a^*b)^*$.

Example

$S = \mathcal{PT}_n$, $E = \{\mathrm{id}_A \mid A \subseteq \{1, \ldots, n\}\}$, $f^+ = \mathrm{id}_{\mathrm{im}(f)}$ and $f^* = \mathrm{id}_{\mathrm{dom}(f)}$.

Ehresmann semigroups

Definition (Lawson, 1991)

Let S be a semigroup and $E \subseteq S$ be a subsemilattice. S is called E-Ehresmann if the following holds:

- For every $a \in S$ there is a minimal $e \in E$ such that $ea = a$ (denoted a^+).
- For every $a \in S$ there is a minimal $e \in E$ such that $ae = a$ (denoted a^*).
- The following identities hold: $(ab)^+ = (ab^+)^+$ and $(ab)^* = (a^*b)^*$.

Example

$S = \mathcal{PT}_n$, $E = \{\mathrm{id}_A \mid A \subseteq \{1, \ldots, n\}\}$, $f^+ = \mathrm{id}_{\mathrm{im}(f)}$ and $f^* = \mathrm{id}_{\mathrm{dom}(f)}$.

Ehresmann semigroups

Definition (Lawson, 1991)

Let S be a semigroup and $E \subseteq S$ be a subsemilattice. S is called E-Ehresmann if the following holds:

- For every $a \in S$ there is a minimal $e \in E$ such that $ea = a$ (denoted a^+).
- For every $a \in S$ there is a minimal $e \in E$ such that $ae = a$ (denoted a^*).
- The following identities hold: $(ab)^+ = (ab^+)^+$ and $(ab)^* = (a^*b)^*$.

Example

$S = \mathcal{PT}_n$, $E = \{\mathrm{id}_A \mid A \subseteq \{1, \ldots, n\}\}$, $f^+ = \mathrm{id}_{\mathrm{im}(f)}$ and $f^* = \mathrm{id}_{\mathrm{dom}(f)}$.

Ehresmann semigroups

Definition (Lawson, 1991)

Let S be a semigroup and $E \subseteq S$ be a subsemilattice. S is called E-Ehresmann if the following holds:

- For every $a \in S$ there is a minimal $e \in E$ such that $ea = a$ (denoted a^+).
- For every $a \in S$ there is a minimal $e \in E$ such that $ae = a$ (denoted a^*).
- The following identities hold: $(ab)^+ = (ab^+)^+$ and $(ab)^* = (a^*b)^*$.

Example

$S = \mathcal{PT}_n$, $E = \{\mathrm{id}_A \mid A \subseteq \{1, \dots, n\}\}$, $f^+ = \mathrm{id}_{\mathrm{im}(f)}$ and $f^* = \mathrm{id}_{\mathrm{dom}(f)}$.

Ehresmann semigroups

Definition (Lawson, 1991)

Let S be a semigroup and $E \subseteq S$ be a subsemilattice. S is called E-Ehresmann if the following holds:

- For every $a \in S$ there is a minimal $e \in E$ such that $ea = a$ (denoted a^+).
- For every $a \in S$ there is a minimal $e \in E$ such that $ae = a$ (denoted a^*).
- The following identities hold: $(ab)^+ = (ab^+)^+$ and $(ab)^* = (a^*b)^*$.

Example

$S = \mathcal{PT}_n$, $E = \{\mathrm{id}_A \mid A \subseteq \{1, \ldots, n\}\}$, $f^+ = \mathrm{id}_{\mathrm{im}(f)}$ and $f^* = \mathrm{id}_{\mathrm{dom}(f)}$.

Ehresmann categories - categories Isomorphism

To every E-Ehresmann S semigroup we can associate an Ehresmann category $\mathcal{C}$. The set of objects is E and the morphisms are in one-to-one correspondence with elements of S.

$$a \in S \quad \Longrightarrow C(a) : a^* \rightarrow a^+$$

$$C(b) \cdot C(a) = \begin{cases} C(ba) & b^* = a^+ \\ \text{undefined} & \text{otherwise} \end{cases}$$

We require this category to be an EI - category.

Ehresmann categories - categories Isomorphism

To every E-Ehresmann S semigroup we can associate an Ehresmann category $\mathcal{C}$. The set of objects is E and the morphisms are in one-to-one correspondence with elements of S.

$$a \in S \quad \Longrightarrow C(a) : a^{*} \to a^{+}$$

$$C(b) \cdot C(a) = \begin{cases} C(ba) & b^{*} = a^{+} \\ \text{undefined} & \text{otherwise} \end{cases}$$

We require this category to be an EI - category.

Ehresmann categories - categories Isomorphism

To every E-Ehresmann S semigroup we can associate an Ehresmann category $\mathcal{C}$. The set of objects is E and the morphisms are in one-to-one correspondence with elements of S.

$$a \in S \quad \Longrightarrow C(a) : a^{*} \to a^{+}$$

$$C(b) \cdot C(a) = \begin{cases} C(ba) & b^{*} = a^{+} \\ \text{undefined} & \text{otherwise} \end{cases}$$

We require this category to be an EI - category.

Ehresmann categories - categories Isomorphism

To every E-Ehresmann S semigroup we can associate an Ehresmann category $\mathcal{C}$. The set of objects is E and the morphisms are in one-to-one correspondence with elements of S.

$$a \in S \quad \Longrightarrow C(a) : a^{*} \to a^{+}$$

$$C(b) \cdot C(a) = \begin{cases} C(ba) & b^{*} = a^{+} \\ \text{undefined} & \text{otherwise} \end{cases}$$

We require this category to be an EI - category.

Ehresmann categories - categories Isomorphism

To every E-Ehresmann S semigroup we can associate an Ehresmann category $\mathcal{C}$. The set of objects is E and the morphisms are in one-to-one correspondence with elements of S.

$$a \in S \quad \Longrightarrow C(a) : a^* \to a^+$$

$$C(b) \cdot C(a) = \begin{cases} C(ba) & b^* = a^+ \\ \text{undefined} & \text{otherwise} \end{cases}$$

We require this category to be an EI - category.

Right restriction

An E-Ehresmann semigroup S is *right restriction* if the following identity holds:

$$ea = a(ea)^*, \quad (e \in E, \quad a \in S)$$

- For $S = \mathcal{PT}_n$, $f\,\mathrm{id}_A$ is restriction of the domain to A and $\mathrm{id}_A f$ is restriction of the image to A. In $\mathcal{PT}_n$, an image restriction can be achieved by domain restriction but not vice versa!

Right restriction

An E-Ehresmann semigroup S is *right restriction* if the following identity holds:

$$ea = a(ea)^*, \quad (e \in E, \quad a \in S)$$

- For $S = \mathcal{PT}_n$, $f\,\mathrm{id}_A$ is restriction of the domain to A and $\mathrm{id}_A f$ is restriction of the image to A. In $\mathcal{PT}_n$, an image restriction can be achieved by domain restriction but not vice versa!

Right restriction

An E-Ehresmann semigroup S is *right restriction* if the following identity holds:

$$ea = a(ea)^*, \quad (e \in E, \quad a \in S)$$

- For $S = \mathcal{PT}_n$, $f\,\mathrm{id}_A$ is restriction of the domain to A and $\mathrm{id}_A\, f$ is restriction of the image to A. In $\mathcal{PT}_n$, an image restriction can be achieved by domain restriction but not vice versa!

Right restriction

An E-Ehresmann semigroup S is *right restriction* if the following identity holds:

$$ea = a(ea)^*, \quad (e \in E, \quad a \in S)$$

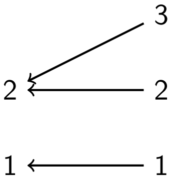
- For $S = \mathcal{PT}_n$, $f\,\mathrm{id}_A$ is restriction of the domain to A and $\mathrm{id}_A f$ is restriction of the image to A. In $\mathcal{PT}_n$, an image restriction can be achieved by domain restriction but not vice versa!

Right restriction

An E-Ehresmann semigroup S is *right restriction* if the following identity holds:

$$ea = a(ea)^*, \quad (e \in E, \quad a \in S)$$

- For $S = \mathcal{PT}_n$, $f\,\mathrm{id}_A$ is restriction of the domain to A and $\mathrm{id}_A f$ is restriction of the image to A. In $\mathcal{PT}_n$, an image restriction can be achieved by domain restriction but not vice versa!

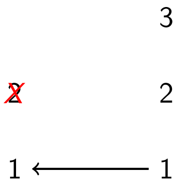


Right restriction

An E-Ehresmann semigroup S is *right restriction* if the following identity holds:

$$ea = a(ea)^*, \quad (e \in E, \quad a \in S)$$

- For $S = \mathcal{PT}_n$, $f\,\mathrm{id}_A$ is restriction of the domain to A and $\mathrm{id}_A\,f$ is restriction of the image to A. In $\mathcal{PT}_n$, an image restriction can be achieved by domain restriction but not vice versa!

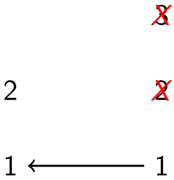


Right restriction

An E-Ehresmann semigroup S is *right restriction* if the following identity holds:

$$ea = a(ea)^*, \quad (e \in E, \quad a \in S)$$

- For $S = \mathcal{PT}_n$, $f\,\mathrm{id}_A$ is restriction of the domain to A and $\mathrm{id}_A f$ is restriction of the image to A. In $\mathcal{PT}_n$, an image restriction can be achieved by domain restriction but not vice versa!

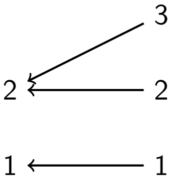


Right restriction

An E-Ehresmann semigroup S is *right restriction* if the following identity holds:

$$ea = a(ea)^*, \quad (e \in E, \quad a \in S)$$

- For $S = \mathcal{PT}_n$, $f\,\mathrm{id}_A$ is restriction of the domain to A and $\mathrm{id}_A f$ is restriction of the image to A. In $\mathcal{PT}_n$, an image restriction can be achieved by domain restriction but not vice versa!

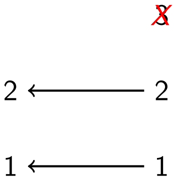


Right restriction

An E-Ehresmann semigroup S is *right restriction* if the following identity holds:

$$ea = a(ea)^*, \quad (e \in E, \quad a \in S)$$

- For $S = \mathcal{PT}_n$, $f\,\mathrm{id}_A$ is restriction of the domain to A and $\mathrm{id}_A f$ is restriction of the image to A. In $\mathcal{PT}_n$, an image restriction can be achieved by domain restriction but not vice versa!

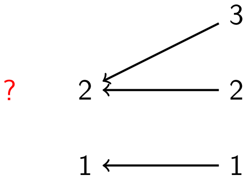


Right restriction

An E-Ehresmann semigroup S is *right restriction* if the following identity holds:

$$ea = a(ea)^*, \quad (e \in E, \quad a \in S)$$

- For $S = \mathcal{PT}_n$, $f\,\mathrm{id}_A$ is restriction of the domain to A and $\mathrm{id}_A f$ is restriction of the image to A. In $\mathcal{PT}_n$, an image restriction can be achieved by domain restriction but not vice versa!



Simple modules

Let S be a finite, E-Ehresmann and right restriction semigroup with corresponding EI-Ehresmann category. Let $a, b \in S$ then $a\mathcal{L}b$ if they generate the same left principal ideal: $S^1a = S^1b$. Let L be a regular $\mathcal{L}$-class (contains an idempotent $e \in L$). Then we can construct as above an $\mathbb{C}S$-module $\mathbb{C}L \otimes V$ (where V is a simple $\mathbb{C}G_e$-module).

Theorem (Margolis & IS)

$\mathbb{C}L \bigotimes V$ is a simple $\mathbb{C}S$-module and every simple module is of this form.

Simple modules

Let S be a finite, E-Ehresmann and right restriction semigroup with corresponding EI-Ehresmann category. Let $a, b \in S$ then $a\mathcal{L}b$ if they generate the same left principal ideal: $S^1a = S^1b$. Let L be a regular $\mathcal{L}$-class (contains an idempotent $e \in L$). Then we can construct as above an $\mathbb{C}S$-module $\mathbb{C}L \otimes V$ (where V is a simple $\mathbb{C}G_e$-module).

Theorem (Margolis & IS)

$\mathbb{C}L \bigotimes V$ is a simple $\mathbb{C}S$-module and every simple module is of this form.

Simple modules

Let S be a finite, E-Ehresmann and right restriction semigroup with corresponding EI-Ehresmann category. Let $a, b \in S$ then $a\mathcal{L}b$ if they generate the same left principal ideal: $S^1a = S^1b$. Let L be a regular $\mathcal{L}$-class (contains an idempotent $e \in L$). Then we can construct as above an $\mathbb{C}S$-module $\mathbb{C}L \otimes V$ (where V is a simple $\mathbb{C}G_e$-module).

Theorem (Margolis & IS)

$\mathbb{C}L \bigotimes V$ is a simple $\mathbb{C}S$-module and every simple module is of this form.

Simple modules

Let S be a finite, E-Ehresmann and right restriction semigroup with corresponding EI-Ehresmann category. Let $a, b \in S$ then $a\mathcal{L}b$ if they generate the same left principal ideal: $S^1 a = S^1 b$. Let L be a regular $\mathcal{L}$-class (contains an idempotent $e \in L$). Then we can construct as above an $\mathbb{C}S$-module $\mathbb{C}L \otimes V$ (where V is a simple $\mathbb{C}G_e$-module).

Theorem (Margolis & IS)

$\mathbb{C}L \bigotimes V$ is a simple $\mathbb{C}S$-module and every simple module is of this form.

Ind. Projective modules

Let S be a finite, E-Ehresmann and right restriction semigroup with corresponding EI-Ehresmann category. Let $a, b \in S$ then $a\widetilde{\mathcal{L}}_E b$ if $a^* = b^*$. Let $\tilde{L}$ be an $\widetilde{\mathcal{L}}_E$-class (which contains precisely one idempotent $e \in E$). Then we can construct as above an $\mathbb{C}S$-module $\mathbb{C}\tilde{L} \otimes V$ (where V is a simple $\mathbb{C}G_e$-module).

Theorem (Margolis & IS)

$\mathbb{C}\tilde{L} \bigotimes V$ is an ind. projective $\mathbb{C}S$-module and every ind. projective module is of this form.

Ind. Projective modules

Let S be a finite, E-Ehresmann and right restriction semigroup with corresponding EI-Ehresmann category. Let $a, b \in S$ then $a\widetilde{\mathcal{L}}_E b$ if $a^* = b^*$. Let $\tilde{L}$ be an $\widetilde{\mathcal{L}}_E$-class (which contains precisely one idempotent $e \in E$). Then we can construct as above an $\mathbb{C}S$-module $\mathbb{C}\widetilde{L} \otimes V$ (where V is a simple $\mathbb{C}G_e$-module).

Theorem (Margolis & IS)

$\mathbb{C}\widetilde{L} \bigotimes V$ is an ind. projective $\mathbb{C}S$-module and every ind. projective module is of this form.

Ind. Projective modules

Let S be a finite, E-Ehresmann and right restriction semigroup with corresponding EI-Ehresmann category. Let $a, b \in S$ then $a\widetilde{\mathcal{L}}_E b$ if $a^* = b^*$. Let $\tilde{L}$ be an $\widetilde{\mathcal{L}}_E$-class (which contains precisely one idempotent $e \in E$). Then we can construct as above an $\mathbb{C}S$-module $\mathbb{C}\widetilde{L} \otimes V$ (where V is a simple $\mathbb{C}G_e$-module).

Theorem (Margolis & IS)

$\mathbb{C}\widetilde{L} \bigotimes V$ is an ind. projective $\mathbb{C}S$-module and every ind. projective module is of this form.

Ind. Projective modules

Let S be a finite, E-Ehresmann and right restriction semigroup with corresponding EI-Ehresmann category. Let $a, b \in S$ then $a\widetilde{\mathcal{L}}_E b$ if $a^* = b^*$. Let $\tilde{L}$ be an $\widetilde{\mathcal{L}}_E$-class (which contains precisely one idempotent $e \in E$). Then we can construct as above an $\mathbb{C}S$-module $\mathbb{C}\widetilde{L} \otimes V$ (where V is a simple $\mathbb{C}G_e$-module).

Theorem (Margolis & IS)

$\mathbb{C}\widetilde{L} \bigotimes V$ is an ind. projective $\mathbb{C}S$-module and every ind. projective module is of this form.

Final remarks

- The class of all finite E-Ehresmann right restriction semigroup with corresponding EI-Ehresmann category is a pseudovariety $\mathcal{V}$ of bi-unary semigroups.
  (pseudovariety - closed under subalgebra, homomorphic image and ***finite*** direct products).
- Clearly, every $(2,1,1)$-subalgebra of $\mathcal{PT}_n$ is in $\mathcal{V}$.

Question

Is every element of $\mathcal{V}$ a (2,1,1)-subalgebra of $\mathcal{PT}_n$?

Proposition (Margolis & IS)

*If $S \in \mathcal{V}$ and **regular** then it is embeddable in $\mathcal{PT}_n$ as a bi-unary semigroup.*

Final remarks

- The class of all finite E-Ehresmann right restriction semigroup with corresponding EI-Ehresmann category is a pseudovariety $\mathcal{V}$ of bi-unary semigroups.
  (pseudovariety - closed under subalgebra, homomorphic image and ***finite*** direct products).
- Clearly, every $(2,1,1)$-subalgebra of $\mathcal{PT}_n$ is in $\mathcal{V}$.

Question

Is every element of $\mathcal{V}$ a (2,1,1)-subalgebra of $\mathcal{PT}_n$?

Proposition (Margolis & IS)

*If $S \in \mathcal{V}$ and **regular** then it is embeddable in $\mathcal{PT}_n$ as a bi-unary semigroup.*

Final remarks

- The class of all finite E-Ehresmann right restriction semigroup with corresponding EI-Ehresmann category is a pseudovariety $\mathcal{V}$ of bi-unary semigroups.
  (pseudovariety - closed under subalgebra, homomorphic image and ***finite*** direct products).
- Clearly, every $(2,1,1)$-subalgebra of $\mathcal{PT}_n$ is in $\mathcal{V}$.

Question

Is every element of $\mathcal{V}$ a (2,1,1)-subalgebra of $\mathcal{PT}_n$?

Proposition (Margolis & IS)

If $S \in \mathcal{V}$ and ***regular*** *then it is embeddable in $\mathcal{PT}_n$ as a bi-unary semigroup.*

Final remarks

- The class of all finite E-Ehresmann right restriction semigroup with corresponding EI-Ehresmann category is a pseudovariety $\mathcal{V}$ of bi-unary semigroups.
  (pseudovariety - closed under subalgebra, homomorphic image and ***finite*** direct products).
- Clearly, every $(2,1,1)$-subalgebra of $\mathcal{PT}_n$ is in $\mathcal{V}$.

Question

Is every element of $\mathcal{V}$ a (2,1,1)-subalgebra of $\mathcal{PT}_n$?

Proposition (Margolis & IS)

*If $S \in \mathcal{V}$ and **regular** then it is embeddable in $\mathcal{PT}_n$ as a bi-unary semigroup.*

Final remarks

- The class of all finite E-Ehresmann right restriction semigroup with corresponding EI-Ehresmann category is a pseudovariety $\mathcal{V}$ of bi-unary semigroups.
  (pseudovariety - closed under subalgebra, homomorphic image and ***finite*** direct products).
- Clearly, every $(2,1,1)$-subalgebra of $\mathcal{PT}_n$ is in $\mathcal{V}$.

Question

Is every element of $\mathcal{V}$ a (2,1,1)-subalgebra of $\mathcal{PT}_n$?

Proposition (Margolis & IS)

If $S \in \mathcal{V}$ and ***regular*** *then it is embeddable in $\mathcal{PT}_n$ as a bi-unary semigroup.*

Thank you!