Groupoid models and C^*-algebras of diagrams of groupoid correspondences

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Étale groupoid correspondences – the definition

Definition (étale groupoid correspondence)

Let $\mathcal{G}$ and $\mathcal{H}$ be étale groupoids.
An étale groupoid correspondence $\mathcal{G} \leftarrow \mathcal{H}$ is
a locally (quasi)compact space Z with commuting actions
$\mathcal{G} \curvearrowright Z \curvearrowleft \mathcal{H}$, such that

- $\sigma\colon Z \to \mathcal{H}^{(0)}$ is a local homeomorphism, and
- the right $\mathcal{H}$-action on Z is free and proper.

A groupoid correspondence is proper/tight
if the map $Z/\mathcal{H} \to \mathcal{G}^{(0)}$ induced by ϱ is proper/a homeomorphism.

Examples of groupoid correspondences

- A (proper) groupoid correspondence on a space X is a (proper) topological correspondence on X.
- A proper groupoid correspondence on a group G is a self-similarity of G.
- Examples of self-similar groups come from injective group homomorphisms and iterated monodromy groups of self-coverings.
- A proper groupoid correspondence on $V \rtimes G$ for an action of a group G on a discrete set V is almost a self-similar graph with vertex set V (Exel–Pardo). We get the slightly more general self-similar groupoid actions on graphs by Laca–Raeburn–Ramagge–Whittaker.

Self-similarities of groups

- Let G and H be groups and let Z be a groupoid correspondence $G \leftarrow H$.
- $\sigma\colon Z \to H^{(0)} = \{\mathrm{pt}\}$ local homeomorphism $\implies Z$ discrete
- Since the right H-action on Z is free and proper, $Z \cong A \times H$ with $(a, h_1) \cdot h_2 = (a, h_1 \cdot h_2)$. Here A may be taken to be a fundamental domain or Z/H.
- Must have $g \cdot (a, h) = (g \cdot a, g|_a \cdot h)$ for a left G-action on A and a map $G \times A \to H$, $(g, a) \mapsto g|_a$.
- $g_1 \cdot (g_2 \cdot (a, h)) = (g_1 \cdot g_2) \cdot (a, h) \iff (g_1 \cdot g_2)|_a = g_1|_{g_2 \cdot a} \cdot g_2|_a$.
- The groupoid correspondence is proper $\iff$ A is finite. If $G = H$, then this is a self-similarity of G. Nekrashevych calls Z a "covering permutational bimodule".
- The groupoid correspondence $G \leftarrow H$ is tight $\iff$ $|A| = 1 \iff Z \cong H$ as a right H-space. Get a group homomorphism $G \to H$.

Examples of self-similar groups

- The first self-similar groups were counterexamples, such as amenable groups of intermediate growth.
- They occur naturally in fractal dynamics, as iterated monodromy groups of self-coverings of topological spaces.
- Self-similarities have also been used, without using this language, to build C*-algebras:

Example (Stammeier, 2015)

An injective group homomorphism $\varphi\colon G \hookrightarrow H$ gives a groupoid correspondence $H \leftarrow G$ with $Z = H$, left H-action by translation, right G-action $h \cdot g := h\varphi(g)$.
(On C*-algebras of irreversible algebraic dynamical systems)

Example (Cuntz–Vershik, 2013)

If G and H are Abelian, $\varphi\colon G \hookrightarrow H$ is equivalent to a surjective group homomorphism $\hat{\varphi}\colon \hat{H} \twoheadrightarrow \hat{G}$.
(C*-algebras associated with endomorphisms and polymorphisms of compact abelian groups)

Iterated monodromy groups

$\mathcal{M}$	space
$\mathcal{M}_1$	open subset in $\mathcal{M}$
p	finite covering map $p\colon \mathcal{M}_1 \to \mathcal{M}$
t	base point in $\mathcal{M}$
G	$\pi_1(\mathcal{M}, t)$
Z	$\{$homotopy classes of paths in $\mathcal{M}$ from t to a point in $p^{-1}(t)\}$
γ	$\gamma \in G$
ζ	$\zeta \in Z$
$\zeta \cdot \gamma$	the usual path concatenation
$\gamma \cdot \zeta$	the concatenation of ζ with the unique p-lift of γ starting at the end point of ζ

Lemma

The right G-action on Z is free.
$Z/G \cong p^{-1}(t)$ through the end point map.

An example (Nekrashevych, Self-Similar Groups, 5.2.2)

- $p\colon \mathbb{C} \setminus \{0, 1, -1\} \to \mathbb{C} \setminus \{0, -1\}$, $z \mapsto z^2 - 1$
- We have removed the critical value -1 of $z^2 - 1$ on $\mathbb{C}$ and its orbit $\{0\}$ to make p a covering.
- Let $t = (1 - \sqrt{5})/2 \in (-1, 0)$. Then $p^{-1}(t) = \{\pm t\}$.
- G is generated freely by the loops around -1 and 0, denoted a and b.
- Let x_0 be the constant path $t \to t$.
- Let x_1 be a path $t \to -t$ through the upper half plane.
- Then $Z = x_0 G \sqcup x_1 G$.
- The left action of the free group G on Z is determined by $ax_0 = x_1 b$, $ax_1 = x_0$, $bx_0 = x_0 a$, $bx_1 = x_1$.

Self-similar actions on graphs

- Let $\mathcal{G} = \mathcal{H} = V \rtimes \Gamma$ for a group Γ and a discrete Γ-set V.
- Let Z be a groupoid correspondence $\mathcal{G} \leftarrow \mathcal{H}$.
- An action of $\mathcal{G} = \mathcal{H}$ is an action of Γ
  with an equivariant anchor map $Z \to V$.
- Z is discrete and of the form $A \times \Gamma$,
  $(a, g) \cdot h = (a, g \cdot h)$, $g \cdot (a, h) = (g \cdot a, g|_a \cdot h)$
- $\sigma(a, g) = g^{-1} \cdot \sigma(a)$, $\varrho(a, g) = \varrho(a)$ for maps $\sigma, \varrho \colon A \rightrightarrows V$.
- Must have $\sigma(g \cdot a) = g|_a \cdot \sigma(a)$, $\varrho(g \cdot a) = g \cdot \varrho(a)$.
- proper groupoid correspondence $\iff$ $\varrho \colon A \to V$ finite-to-one
- tight groupoid correspondence $\iff$ $\varrho \colon A \to V$ bijective

Remark

When combining graph algebras and Nekrashevych's algebras of self-similar groups, the right generality seems a discrete groupoid $\mathcal{G}$ with a groupoid correspondence $\mathcal{G} \leftarrow \mathcal{G}$.

From groupoid to C*-correspondences

Definition

Let A, B be C*-algebras.
An A, B-correspondence or correspondence $A \leftarrow B$ is a Hilbert B-module $\mathcal{E}$ with a nondegenerate *-homomorphism $A \to \mathbb{B}(\mathcal{E})$.
A correspondence is proper if $A \to \mathbb{K}(\mathcal{E})$.

- Let $\mathcal{G}, \mathcal{H}$ be groupoids and Z a correspondence $\mathcal{G} \leftarrow \mathcal{H}$.
- $\mathfrak{S}(Z)$: linear span of $C_c(U)$ for Hausdorff open $U \subseteq Z$
- Make $\mathfrak{S}(Z)$ a bimodule over $\mathfrak{S}(\mathcal{G})$ and $\mathfrak{S}(\mathcal{H})$ by

$$(a * \xi)(z) := \sum_{g \in \mathcal{G}^{\varrho(z)}} a(g)\xi(g^{-1}z),$$

$$(\xi * b)(z) := \sum_{h \in \mathcal{H}^{\sigma(z)}} \xi(zh)b(h^{-1})$$

- Define an inner product on $\mathfrak{S}(Z)$ by $\langle \xi_1 \mid \xi_2 \rangle :=$ "$\xi_1^* * \xi_2$":

$$\langle \xi_1 \mid \xi_2 \rangle(h) := \sum_{z \in Z : \sigma(z) = r(h)} \overline{\xi_1(z)}\xi_2(z \cdot h).$$

From groupoid to C^*-correspondences II

Proposition

A (proper) groupoid correspondence $Z\colon \mathcal{G} \leftarrow \mathcal{H}$
induces a (proper) C^**-correspondence* $C^*(Z)\colon C^*(\mathcal{G}) \leftarrow C^*(\mathcal{H})$.

Proof steps.

- The inner product on $\mathfrak{S}(Z)$ is positive definite.
- We complete $\mathfrak{S}(Z)$ in the norm $\|\xi\| := \|\langle \xi \mid \xi \rangle\|^{1/2}$ to a Hilbert module $C^*(Z)$ over $C^*(\mathcal{H})$.
- The left action of $\mathfrak{S}(\mathcal{G})$ on $\mathfrak{S}(Z)$ extends to a nondegenerate *-homomorphism $C^*(\mathcal{G}) \to \mathbb{B}(C^*(Z))$.
- $C^*(\mathcal{G}) \to \mathbb{K}(C^*(Z)) \iff Z$ is proper □

The Toeplitz algebra

A, D C*-algebras

$\mathcal{E}$ A, A-correspondence

φ *-homomorphism $A \to D$

S linear map $\mathcal{E} \to D$

Definition (Toeplitz representation)

Call (φ, S) a Toeplitz representation of $\mathcal{E}$ if

- $S(x)^* S(y) = \varphi(\langle x \mid y \rangle)$ for $x, y \in \mathcal{E}$
- $S(ax) = \varphi(a) S(x)$ and $S(xa) = S(x)\varphi(a)$ for $a \in A$, $x \in \mathcal{E}$

Definition (Toeplitz algebra)

The Toeplitz algebra of $\mathcal{E}$ is a C*-algebra $\mathcal{T}_{\mathcal{E}}$ such that *-homomorphisms $\mathcal{T}_{\mathcal{E}} \to D$ are in natural bijection with Toeplitz representations of $\mathcal{E}$ in D.

The Cuntz–Pimsner algebra

Let $\mathcal{E}$ with left action $\psi\colon A \to \mathbb{B}(\mathcal{E})$ be an A, A-correspondence.

Lemma

Let (φ, S) be a Toeplitz representation.
*There is a unique *-homomorphism $\varphi_1\colon \mathbb{K}(\mathcal{E}) \to D$ with $\varphi_1(|x\rangle\langle y|) = S(x)S(y)^*$ for $x, y \in \mathcal{E}$.*

Definition

Let $J \subseteq A$ be an ideal with $\varphi(J) \subseteq \mathbb{K}(\mathcal{E})$.
A Toeplitz representation (φ, S) is Cuntz–Pimsner covariant on J if $\varphi(a) = \varphi_1(\psi(a))$ for all $a \in J$.

Definition (relative Cuntz–Pismner algebra)

The Cuntz–Pismner algebra of $\mathcal{E}$ relative to J is a C*-algebra $\mathcal{O}_{\mathcal{E},J}$ such that *-homomorphisms $\mathcal{O}_{\mathcal{E},J} \to D$ are in natural bijection with Toeplitz representations of $\mathcal{E}$ in D that are Cuntz–Pimsner covariant on J.

Graph algebras as Cuntz–Pimsner algebras

Example (Cuntz algebras)

Let $A = \mathbb{C}$, $\mathcal{E} = \mathbb{C}^n$ for $n \in \mathbb{N}$. The Cuntz–Pimsner algebra for $\mathcal{E}$ (relative to A) is, by definition, the Cuntz algebra $\mathcal{O}_n$.

Example (Graph algebras)

- Let $r, s \colon E \rightrightarrows V$ be a directed graph.
- View it as a correspondence on the discrete groupoid V.
- Let $\mathcal{E}$ be the induced C^*-correspondence $\mathrm{C}_0(V) \leftarrow \mathrm{C}_0(V)$.
- The subset of regular vertices is

$$V_{\mathrm{reg}} := \{v \in V : r^{-1}(v) \text{ is finite and non-empty}\}.$$

- The graph C^*-algebra of the directed graph $r, s \colon E \rightrightarrows V$ is the Cuntz–Pimsner algebra of $\mathcal{E}$ relative to $\mathrm{C}_0(V_{\mathrm{reg}})$.
- In the following, will assume $V = V_{\mathrm{reg}}$.

The Cuntz–Pimsner algebra for a groupoid correspondence

- Let $\mathcal{G}$ be an étale, locally compact groupoid.
- Let Z be a proper étale, locally compact groupoid correspondence $\mathcal{G} \leftarrow \mathcal{G}$.
- We have defined a proper C^*-correspondence $\mathrm{C}^*(Z)\colon \mathrm{C}^*(\mathcal{G}) \leftarrow \mathrm{C}^*(\mathcal{G})$.
- We call its (absolute) Cuntz–Pimsner algebra the Cuntz–Pimsner algebra of the groupoid correspondence.

Examples of Cuntz–Pimsner algebras

Example (Topological graph algebras)

Let $\varrho, \sigma \colon Z \rightrightarrows X$ be a groupoid correspondence on a space X.
Assume ϱ to be surjective and proper.
The Cuntz–Pimsner algebra of this groupoid correspondence is Katsura's topological graph algebra.

Example (Nekrashevych's C*-algebra of a self-similar group)

Let G be a self-similar group with permutational bimodule Z, viewed as a proper groupoid correspondence $G \leftarrow G$.
Its Cuntz–Pimsner algebra is Nekrashevych's C*-algebra.

Example (Exel–Pardo's C*-algebras of self-similar graphs)

A proper groupoid correspondence $Z \colon V \ltimes \Gamma \leftarrow V \ltimes \Gamma$ for a discrete set V and an action of a discrete group Γ on V with surjective $r \colon Z \to V$ is equivalent to a self-similar graph as generalised by Laca–Raeburn–Ramagge–Whittaker. The Cuntz–Pimsner algebra of the groupoid correspondence is the C*-algebra studied before.

The groupoid model of a groupoid correspondence

- Let $\mathcal{X}\colon \mathcal{G} \leftarrow \mathcal{G}$ be a groupoid correspondence.
- The Cuntz–Pimsner algebra of $\mathrm{C}^*(\mathcal{X})\colon \mathrm{C}^*(\mathcal{G}) \leftarrow \mathrm{C}^*(\mathcal{G})$ is isomorphic to a groupoid C^*-algebra.
- We describe the underlying groupoid concretely, and by a universal property.
- The universal property describes actions of the groupoid model on locally compact spaces.
- Both descriptions use the universal action of a groupoid correspondence.
- This is an analogue of the space of complete histories for a topological correspondence.
- Both descriptions use the composition of groupoid correspondences.

Composition of groupoid correspondences

- Let $\mathcal{G}, \mathcal{H}, \mathcal{K}$ be étale groupoids.
- Let $\mathcal{X}\colon \mathcal{G} \leftarrow \mathcal{H}$ and $\mathcal{Y}\colon \mathcal{H} \leftarrow \mathcal{K}$ be groupoid correspondences.
- $\mathcal{X} \times_{\mathcal{H}^{(0)}} \mathcal{Y} := \{(x, y) \in \mathcal{X} \times \mathcal{Y} : \sigma(x) = \varrho(y)\}$.
- Let $\mathcal{G}$ and $\mathcal{K}$ act on $\mathcal{X} \times_{\mathcal{H}^{(0)}} \mathcal{Y}$ by $g \cdot (x, y) := (g \cdot x, y)$, $(x, y) \cdot k := (x, y \cdot k)$.
- Let $\mathcal{H}$ act on $\mathcal{X} \times_{\mathcal{H}^{(0)}} \mathcal{Y}$ by $h \cdot (x, y) := (x \cdot h^{-1}, h \cdot y)$.

Lemma

The $\mathcal{G}, \mathcal{K}$-actions on $\mathcal{X} \times_{\mathcal{H}^{(0)}} \mathcal{Y}$ descend to commuting actions on the $\mathcal{H}$-orbit space $\mathcal{X} \times_{\mathcal{H}^{(0)}} \mathcal{Y}/\mathcal{H}$, making it a groupoid correspondence $\mathcal{X} \circ \mathcal{Y}\colon \mathcal{G} \leftarrow \mathcal{K}$.

Proof.

The $\mathcal{H}$-action on $\mathcal{X}$ is free and proper.
The projection $\mathcal{X} \times_{\mathcal{H}^{(0)}} \mathcal{Y} \to \mathcal{X}$ is $\mathcal{H}$-equivariant. $\square$

Associativity and unit correspondences

Lemma

Consider a chain of three composable groupoid correspondences

$$\mathcal{G} \xleftarrow{\mathcal{X}} \mathcal{H} \xleftarrow{\mathcal{Y}} \mathcal{K} \xleftarrow{\mathcal{Z}} \mathcal{L}.$$

Then there is a natural isomorphism of groupoid correspondences $(\mathcal{X} \circ \mathcal{Y}) \circ \mathcal{Z} \cong \mathcal{X} \circ (\mathcal{Y} \circ \mathcal{Z})$.

Definition (unit correspondence)

Let $\mathcal{G}$ be an étale groupoid. Equip the arrow space $\mathcal{G}$ with the actions of $\mathcal{G}$ by left and right multiplication. This is called the unit correspondence on $\mathcal{G}$.

Lemma

$\mathcal{G} \circ \mathcal{X} \cong \mathcal{X}$ *and* $\mathcal{X} \circ \mathcal{H} \cong \mathcal{X}$ *by multiplication.*

Remark

This makes $\mathcal{X}^{\circ n}$ for $n \in \mathbb{N}$ well defined and well behaved.

Actions of a groupoid correspondence

- Let $\mathcal{X}\colon \mathcal{G} \leftarrow \mathcal{G}$ be a groupoid correspondence.
- Let Y be a locally compact space with a left action of $\mathcal{G}$.
- Define $\mathcal{X} \circ Y$ as above, as the orbit space of the $\mathcal{G}$-action on $\mathcal{X} \times_{\mathcal{G}^{(0)}} Y$ defined by $g \cdot (x, y) := (xg^{-1}, gy)$. Equip $\mathcal{X} \circ Y$ with the obvious left $\mathcal{G}$-action $g \cdot [(x, y)] := [(g \cdot x, y)]$.
- An action of the groupoid correspondence $\mathcal{X}\colon \mathcal{G} \leftarrow \mathcal{G}$ is a locally compact space Y with a $\mathcal{G}$-action and a $\mathcal{G}$-equivariant homeomorphism $\alpha\colon \mathcal{X} \circ Y \xrightarrow{\sim} Y$.
- Let Y_1 and Y_2 be spaces with such actions.
A continuous map $f\colon Y_1 \to Y_2$ is equivariant if it is $\mathcal{G}$-equivariant and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{X} \circ Y_1 & \xrightarrow{\alpha_1} & Y_1 \\ {\scriptstyle \mathrm{id}_{\mathcal{X}} \circ f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{X} \circ Y_2 & \xrightarrow{\alpha_2} & Y_2 \end{array}$$

Universal property of the groupoid model

Definition

A groupoid model for the groupoid correspondence $\mathcal{X}\colon \mathcal{G} \leftarrow \mathcal{G}$ is a groupoid $\mathcal{H}$ such that for any locally compact space Y, there is a natural bijection $\{\mathcal{H}\text{-actions on } Y\} \cong \{\mathcal{X}\text{-actions on } Y\}$.
Naturality: for a continuous map $f\colon Y_1 \to Y_2$
$\mathcal{H}$-equivariant $\iff$ $\mathcal{X}$-equivariant

Lemma

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be étale groupoids with a natural bijection between actions of $\mathcal{H}_1$ and $\mathcal{H}_2$ on Y for all locally compact spaces Y. Then $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic as topological groupoids.
(*And the natural bijection comes from this isomorphism.*)

Proposition

The groupoid model of a groupoid correspondence exists.
It is unique up to isomorphism.

Universal action of a groupoid correspondence

Definition

A universal action of a groupoid correspondence $\mathcal{X}\colon \mathcal{G} \leftarrow \mathcal{G}$ is a terminal object in the category of $\mathcal{X}$-actions.

Lemma

The category of $\mathcal{H}$-actions for an étale groupoid $\mathcal{H}$ has the canonical action on $\mathcal{H}^{(0)}$ as a terminal object.

Lemma

The object space of the groupoid model of a groupoid correspondence must be a universal action of it.

An example

$\mathcal{G}$ pt: one-object one-arrow groupoid

$\mathcal{X}$ $\{1,2\}$ correspondence pt $\leftarrow$ pt with two points

Y locally compact space

- An $\mathcal{X}$-action on Y is equivalent to a homeomorphism

$$\alpha\colon \{1,2\} \times Y \xrightarrow{\sim} Y.$$

- Iteration gives homeomorphisms $\{1,2\}^n \times Y \xrightarrow{\sim} Y$.
- They give continuous maps $Y \to \{1,2\}^n$ for all $n \in \mathbb{N}$.
- These combine to a continuous map $Y \to \{1,2\}^{\mathbb{N}}$.
- The obvious homeomorphism $\{1,2\} \times \{1,2\}^{\mathbb{N}} \xrightarrow{\sim} \{1,2\}^{\mathbb{N}}$ is the universal action of $\mathcal{X}$.
- $\{1,2\}^{\mathbb{N}}$ is the space of complete histories for the topological correspondence $\mathcal{X}$.

An analogue of the space of complete histories

- Let $\mathcal{X}\colon \mathcal{G} \leftarrow \mathcal{G}$ be a groupoid correspondence.
- For $n \in \mathbb{N}$, form the n-fold composition $\mathcal{X}^{\circ n}\colon \mathcal{G} \leftarrow \mathcal{G}$.
- The map $\mathcal{X}^{\circ n+1} \to \mathcal{X}^{\circ n}$, $[(x_0, \dots, x_n)] \to [(x_0, \dots, x_{n-1})]$, is not well defined.
- $(x_0, \dots, x_n) \sim (x_0 g_0^{-1}, g_0 x_1 g_1^{-1}, \dots, g_{n-2} x_{n-1} g_{n-1}^{-1}, g_{n-1} x_n)$
- The map $\mathcal{X}^{\circ n+1}/\mathcal{G} \to \mathcal{X}^{\circ n}/\mathcal{G}$, $[(x_0, \dots, x_n)] \to [(x_0, \dots, x_{n-1})]$, is well defined.

Definition

Let $\Omega := \varprojlim \mathcal{X}^{\circ n}/\mathcal{G}$.

Lemma

There is a well defined action of $\mathcal{G}$ on Ω by $g \cdot [(x_0, x_1, x_2, \dots)] := [(g \cdot x_0, x_1, x_2, \dots)]$.
The map $\mathcal{X} \times_{\mathcal{G}^{(0)}} \Omega \to \Omega$, $(x_0, [(x_1, x_2, \dots)]) \mapsto [(x_0, x_1, x_2, \dots)]$, induces a canonical homeomorphism $\mathcal{X} \circ \Omega \xrightarrow{\sim} \Omega$.