Self-similar graphs and their algebras.

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Outline

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The reason to consider this construction was to study two different kind of algebras under the same umbrella.

First, a construction introduced by V. Nekrashevych, in order to study self-similar groups (group actions).

$X = \{a_1, \ldots, a_n\}$ finite alphabet. G group acting on the n-rooted tree on X by automorphisms.

$g(a_i a_j) = g(a_i) \cdot g_{|a_i}(a_j)$ defines a 1-cocycle on G:

$$\begin{array}{ccc} G \times X & \to & G \\ (g, a_i) & \mapsto & g_{|a_i} \end{array}$$

with $gh_{|a} = g_{|ha} \cdot h_{|a}$ and $g_{|ab} = (g_{|a})_{|b}$.

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Definition

Define $\mathcal{O}_{(G,X)}$ to be the universal C^*-algebra generated by isometries $\{s_{a_1}, \ldots, s_{a_n}\}$ and unitaries $\{u_g \mid g \in G\}$ satisfying the relations:

(i) $s_{a_i}^* s_{a_j} = \delta_{i,j}$ for all $1 \leq i, j \leq n$.

(ii) $1 = \sum_{i=1}^{n} s_{a_i} s_{a_i}^*$

(iii) $u_g s_{a_i} = s_{g(a_i)} u_{g|_{a_i}}$ for all $g \in G$ and $1 \leq i \leq n$.

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Now, the following facts holds:

1. $\mathcal{O}_{(G,X)}$ is separable, and there are conditions for being nuclear and in the UCT class.
2. They allows to deal with group properties in algebra terms.
3. For any group G, $\mathcal{O}_n \hookrightarrow \mathcal{O}_{(G,X)}$.

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Second, a construction introduced by T. Katsura as an example of a topological graph C^*-algebra representing Kirchberg algebras in the UCT in a combinatorial way.

Definition

Let $N \in \mathbb{N} \cup \{\infty\}$, let $A \in M_N(\mathbb{Z}^+)$ and $B \in M_N(\mathbb{Z})$ be row-finite matrices. Define a set Ω_A by

$$\Omega_A := \{(i,j) \in \{1,2,\ldots,N\} \times \{1,2,\ldots,N\} \mid A_{i,j} \geq 1\}.$$

Fix the following condition:

(0) $\Omega_A(i) \neq \emptyset$ *for all* i, *and* $B_{i,j} = 0$ *for* $(i,j) \notin \Omega_A$.

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(i) $s_{i,j,n}^* s_{i,j,n} = q_j$ for all $(i,j) \in \Omega_A$ and $n \in \mathbb{Z}$.

(ii) $q_i = \sum_{j\in\Omega_A(i)} \sum_{n=1}^{A_{i,j}} s_{i,j,n} s_{i,j,n}^*$ for all i.

(iii) $s_{i,j,n} u_j = s_{i,j,n+A_{i,j}}$ and $u_i s_{i,j,n} = s_{i,j,n+B_{i,j}}$ for all $(i,j) \in \Omega_A$ and $n \in \mathbb{Z}$.

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How to connect both classes of algebras?

Relation (iii) in definition is equivalent to

$$u_i s_{i,j,n} = s_{i,j,\widehat{n}} u_j^k$$

for unique $1 \leq \widehat{n} \leq A_{i,j}$ and $k \in \mathbb{Z}$ such that $n + B_{i,j} = \widehat{n} + kA_{i,j}$.

If E_A is a finite graph, then $u := \sum_{i=1}^{N} u_i$ is a unitary of $\mathcal{O}_{A,B}$.

For any i, j, $us_{i,j,n} = u_i s_{i,j,n}$ and $s_{i,j,n} u = s_{i,j,n} u_j$.

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Action of $\mathbb{Z}$ on E_A: Given (i,j,n) with $1 \leq n \leq A_{ij}$, $l \in \mathbb{Z}$, define $l \cdot (i,j,n) = (i,j,\widehat{n})$ for the unique $1 \leq \widehat{n} \leq A_{i,j}$ and $k \in \mathbb{Z}$ such that $n + lB_{i,j} = \widehat{n} + kA_{i,j}$.

1-cocycle: For the above data,

$$\varphi(l,(i,j,n)) = k = \frac{(n-\widehat{n}) + lB_{ij}}{A_{ij}}.$$

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They should come from a general setting.

Outline

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$\sigma : G \times E \to E$, graph action.

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Let $g \in G$, $a, b \in E^1$ with $d(a) = r(b)$.

$$g \cdot (ab) = (g \cdot a)(\varphi(g, a) \cdot b)$$

The action must satisfy

$$g \cdot x = \varphi(g, a) \cdot x \qquad \forall g \in G, \forall a \in E^1, \forall x \in E^0.$$

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Given (G, E, φ), define $\mathcal{O}_{G,E}$ to be the universal C^*-algebra generated by mutually orthogonal projections $\{p_x \mid x \in E^0\}$, partial isometries $\{s_a \mid a \in E^1\}$ and unitaries $\{u_g \mid g \in G\}$ satisfying the relations:

(i) $\{p_x \mid x \in E^0\} \cup \{s_a \mid a \in E^1\}$ generate $C^*(E)$.

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If the 1-cocycle is $\varphi(g,a) = g$ for all $g \in G, a \in E^1$, then $\mathcal{O}_{G,E} \cong C^*(E) \rtimes G$ (crossed product).

If the 1-cocycle is $\varphi(g,a) = 1$ for all $g \in G, a \in E^1$, then vertices are fixed by the action of G, and $\mathcal{O}_{G,E} \cong C^*(E)$.

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Nekrashevych and Katsura algebras, by the arguments showed before.

Outline

Definition

A *groupoid* $\mathcal{G}$ is an small category in which every homomorphism is an isomorphism. We will denote by $\mathcal{G}^{(0)}$ its set of units, and by $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ the range and source maps $r(\gamma) = \gamma\gamma^*$ and $s(\gamma) = \gamma^*\gamma$.

Definition

A *topological groupoid* is a groupoid endowed with a topology under which multiplication and inversion are continuous maps; in particular, r and s are continuous maps.

Definition

A topological groupoid $\mathcal{G}$ is said to be *étale* if $\mathcal{G}^{(0)}$ is locally compact and Hausdorff in the relative topology, and r (and so s) is a local homeomorphism from $\mathcal{G}$ to $\mathcal{G}^{(0)}$.

Definition

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An étale groupoid $\mathcal{G}$ is said to be *ample* if $\mathcal{G}^{(0)}$ is totally disconnected.

Let $\mathcal{G}$ be an étale groupoid, let $C_c(\mathcal{G})$ the linear span of the set of continuous, complex valued, compactly supported functions on $\mathcal{G}$.

The operation

$$f * g(x) = \sum_{x=yz} f(y)g(z) \text{ and } f^*(x) = \overline{f(x^{-1})}$$

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Definition

The *groupoid* C^**-algebra* $C^*(\mathcal{G})$ is the norm-completion of $C_c(\mathcal{G})$ in a suitable norm.

If X is a locally compact, zero-dimensional and Hausdorff space, and if R is a unital commutative ring, we denote by

$$C_c(X, R)$$

the R-module formed by all locally constant, compactly supported, R-valued functions on X.

Definition

Given an ample groupoid $\mathcal{G}$, and a unital commutative ring R, the *Steinberg algebra* $A_R(\mathcal{G})$, is defined to be the R-algebra obtained by equipping $C_c(\mathcal{G}, R)$ with the convolution product

$$(fg)(\gamma) = \sum_{\gamma=\gamma_1\gamma_2} f(\gamma_1)g(\gamma_2).$$

The groupoid approach lets us to deal with some questions in a unique language.

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If $\mathcal{G}$ is amenable & Hausdorff, then $C^*(\mathcal{G})$ simple iff $\mathcal{G}$ is minimal & essentially principal **[Brown-Clark-Farthing-Sims].**

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Outline

1. The origins
2. Self-similar graphs
3. Groupoids and their algebras
4. The tight groupoid and its C^*-algebra.
5. Algebras of self similar graphs.
6. Further developments

Definition

A semigroup T is an *inverse semigroup* if

- for every x in T, there exists a unique $x^* \in T$, such that $xx^*x = x$ and $x^*xx^* = x^*$,
- there exists a (necessarily unique) element $0 \in T$, called the zero element, such that $x0 = 0x = 0$, for all x in T.

If T is an inverse semigroup, then the set of idempotents of T, $\mathcal{E} = \mathcal{E}(T)$, is a semilattice with ordering $e \leq f$ if and only if $ef = e$, and $e \wedge f = ef$.

This order extends to an order in T, $s \leq t$ if and only if $s = ts^*s = ss^*t$. We denote by $e \perp f$ if and only if $ef = 0$, and $e \Cap f$ if and only if $ef \neq 0$.

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Definition

A filter in $\mathcal{E}$ is a nonempty subset $\eta \subseteq \mathcal{E}$ such that:

1. $0 \notin \eta$,
2. closed under $\wedge$,
3. $f \geq e \in \eta$ implies $f \in \eta$.

We denote the set of filters by $\widehat{\mathcal{E}_0}$.This is a locally compact totally disconnected Hausdorff space when equipped with a cylinder topology.

Definition

A filter η is an *ultrafilter* if it is not properly contained in another filter. We denote $\widehat{\mathcal{E}}_\infty \subseteq \widehat{\mathcal{E}}_0$ the space of ultrafilters.

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The set $\widehat{\mathcal{E}}_{\text{tight}}$ of *tight filters* is the closure of $\widehat{\mathcal{E}}_\infty$ into $\widehat{\mathcal{E}}_0$.

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Definition

We can define a standard partial action of T on $\widehat{\mathcal{E}}_0$ as follows:

1. For each $e \in \mathcal{E}$, $D_e^\beta = \{\eta \in \widehat{\mathcal{E}}_0 : e \in \eta\}$,
2. Given $s \in T$,

$$\begin{aligned} \beta_s : D_{s^*s}^\beta &\longrightarrow D_{ss^*}^\beta \\ \eta &\longrightarrow \beta_s(\eta) = \{f \in \mathcal{E} : f \geq ses^* \text{ for every } e \in \eta\} \end{aligned}$$

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β restricts to an action of T on ultrafilters and on tight filters.

Consider the transformation groupoid $T \times \widehat{\mathcal{E}}_{\text{tight}}$.

The elements are the pairs (s, ω) such that $\omega \in \text{Dom}(s) = D^{\beta}_{s^*s}$.

We fix the germ relation: $(s, \omega) \sim (t, \eta)$ if $\omega = \eta$ and there exists an idempotent $e \leq t, s$ with $\omega \in \text{Dom}(e)$ such that $se = te$.

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Definition

Define $\mathcal{G}_{\text{tight}}(T) = T \times \widehat{\mathcal{E}}_{\text{tight}} / \sim$, with:

1. $d([s,x]) = x$ and $r([s,x]) = \beta_s(x)$,
2. $[s,z] \cdot [t,x] = [st,x]$ if and only if $z = \beta_t(x)$,
3. $[s,x]^{-1} = [s^*, \beta_s(x)]$,
4. $\mathcal{G}^{(0)}_{\text{tight}}(T) = \{[e,x] : e \in \mathcal{E}\} \cong \widehat{\mathcal{E}}_{\text{tight}}$

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Given any $F \subseteq \mathcal{E}$, we say that $Z \subseteq F$ is a cover for F if for every $0 \neq x \in F$ there exists $z \in Z$ such that $zx \neq 0$. Z is cover for $y \in \mathcal{E}$ if it is a cover for $F = \{x \in \mathcal{E} : x \leq y\}$.

A representation $\phi : T \to B$ (where B is a unital C^*-algebra) is *tight* if for every $e \in \mathcal{E}$ and every cover $\{e_1, \ldots, e_n\}$ for e we have

$$\phi(e) = \bigvee_{1 \leq i \leq n} \phi(e_i),$$

where "$\bigvee$" refers to the operation of taking supremum of a commuting set of projections.

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Suppose that A is a C^*-algebra, T an inverse semigroup, and

$$\pi : T \to A^{\sim}$$

is a $*$-representation of T into $A^{\sim}$.

Theorem

If $\pi : T \to A^{\sim}$ is a universal tight representation of T, then

$$A \cong C^*(\mathcal{G}_{\mathit{tight}}(T))$$

In most cases, the same result applies for K any field and A a K-algebra

Theorem

If $\pi : T \to A^\sim$ is a universal tight representation of T, then

$$A \cong A_K(\mathcal{G}_{\text{tight}}(T))$$

Thus, we can characterize various properties of A looking at the properties of $\mathcal{G}_{\text{tight}}(T)$, or equivalently, to the properties enjoyed by the action of T on $\widehat{\mathcal{E}}_{\text{tight}}$

Outline

Fix a $*$-inverse semigroup

$$\mathcal{S}_{G,E} := \{(\alpha, g, \beta) \mid g \in G, \alpha, \beta \in E^* \text{ with } d(\alpha) = gd(\beta)\},$$

with the operation induced by the product of elements in $\mathcal{O}_{G,E}$.

Consider the natural semigroup map

$$\begin{array}{rccc} \pi: & \mathcal{S}_{G,E} & \to & \mathcal{O}_{G,E} \\ & (\alpha, g, \beta) & \mapsto & s_\alpha u_g s_\beta^* \end{array}$$

Proposition

$\pi : \mathcal{S}_{G,E} \to \mathcal{O}_{G,E}$ is the universal tight representation of $\mathcal{S}_{G,E}$.

Theorem

For any field K, here are $*$-isomorphisms

$$\mathcal{O}_{G,E} \cong C^*(\mathcal{G}_{\textit{tight}}(\mathcal{S}_{G,E})) \text{ and } \mathcal{O}^{\textit{alg}}_{G,E}(K) \cong A_K(\mathcal{G}_{\textit{tight}}(\mathcal{S}_{G,E})).$$

We first characterize when $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is Hausdoff. A property that plays a central role is

Definition (Strongly fixed path)

Given $g \in G$, we say that a finite path α is strongly fixed by g if

$$g\alpha = \alpha \text{ and } \varphi(g, \alpha) = 1.$$

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Theorem

TFAE:

1. *For every g in G, there are at most finitely many minimal strongly fixed paths for g.*
2. $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ *is Hausdorff.*

With respect to minimality, we have

Theorem

TFAE:

1. *The matrix A_E is G-irreducible.*
2. *The groupoid $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is minimal.*

To characterize effectiveness, we need an extra definition

Definition (Slack at x)

Given $g \in G$, and $x \in E^0$, we say that g is slack at x, if there is $n \in \mathbb{Z}^+$ such that all finite paths γ with $\text{ran}(\gamma) = x$, and $|\gamma| \geq n$, are strongly fixed by g.

Thus, we get

Theorem

TFAE:

1. *The groupoid $\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E})$ is effective.*
2. (i) *The graph E satisfies Condition (L).*
   (ii) *Given a vertex x, and a group element g fixing every infinite path in $Z(x)$, then necessarily g is slack at x.*

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Theorem

Let $(G, E; \varphi)$ with G amenable, A the adjacency matrix of E, and such that for every g in G, there are at most finitely many minimal strongly fixed paths for g. Then the following are equivalent:

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2. $A_K(\mathcal{G}_{\text{tight}}(\mathcal{S}_{G,E}))$ *is simple.*

Outline

1. The origins
2. Self-similar graphs
3. Groupoids and their algebras
4. The tight groupoid and its C^*-algebra.
5. Algebras of self similar graphs.
6. Further developments

In a sequent (arXiv) work, Exel-P.-Starling extended the definition of self-similar action to arbitrary (countable) graphs:

Via a desingularization process, reduce the problem to row-finite graphs with no sources or sinks.

Under this restriction, developed a tight groupoid representation & characterized properties (except locally contracting).

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Self-similar graphs and their algebras.

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May 28, 2020.