Solution to the Sandwich Classification Problem in arbitrary groups and applications to classical-like groups over arbitrary rings

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Introduction

Let G be a group and F and H subgroups of G. By definition, the mixed commutator subgroup $[F, H]$ of G is the subgroup

$$\text{generated by all commutators } [f, h] = fhf^{-1}h^{-1}$$

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where $f \in F$ and $h \in H$.

$[F, H]$ is called an F-**commutator subgroup** of G.

We summarize our main structural result.

Introduction

Main Structure Theorem

Let G be an arbitrary group and F an arbitrary subgroup of G. Then a subgroup

$$H \text{ of } G \text{ is normalized by } F \iff$$

$$H \text{ is a member of a unique sandwich } S(K)$$

where K ranges over all F-perfect subgroups of G.

The notion of a sandwich $S(K)$ will be defined below.

Introduction

Sandwich classification theorems are established by implementing F in the structure theorem above with either a *structured set of generators or a structured set of generating subgroups.*

We define now the concept of sandwich above.

Cocommutator subgroups

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Let F and J be subgroups of G.
Let K denote the F-commutator group $K = [F, J]$.

We define an

- F-**cocommutator subgroup over** K to be any subgroup H of G such that $H \supseteq K$ and $[F, H] = K$.
  (It follows that H is F-normal.)

Sandwiches

Define

- **S(K)** to be the set of all F-cocommutator subgroups over K.

For the sake of clarity, we sometimes write

- **S(F,K)** in place of $S(K)$.

The concept full F-cocommutator subgroup is defined in the Main Structure Theorem below.

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S(F,K) satisfies the following properties:

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and

- *if $\mathcal{S} \subseteq S(F,K)$ then $\langle \mathcal{S} \rangle \in S(F,K)$.*

Main Structure Theorem, continued

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- *if F is perfect then $S(F, K)$ has an absolutely smallest member, namely K.*

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Moreover, sandwich classification holds:

- *A subgroup*

$$H \text{ of } G \text{ is normalized by } F \iff$$

$$H \text{ is a member of a sandwich } S(F, K)$$

and K is unique.

The example of GL_n

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If R is commutative or more generally quasi-finite, then we can describe more explicitly the sandwiches $S(F, K)$.

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The sandwiches $S(F, K)$ are the sets of groups

$$\{H \mid E_n(R, I) \subseteq H \subseteq CGL_n(R, I)\}, \quad I \text{ an ideal.}$$

Further implementations to classical-like groups

There is a completely analogous result for any classical-like group $G = G(R)$ and $F = E(R)$ when $E(R)$ is perfect
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We want to extend the result above for classical-like groups over quasi-finite rings to classical-like groups over arbitrary rings.

Further implementations to classical-like groups, continued

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there is a further **tacit** *implementation*,

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Clearly the elementary matrices play an important role in describing the F-commutator subgroups K which define the sandwiches $S(F, K)$.

perfect elements and perfect sets of elements

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perfect in F, if $x \in F$ and

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A set X of elements of G is called

perfect in F, if $X \subseteq F$ and

each element of X is perfect in F.

absolutely perfect sets of elements

Lemma

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Recall: An F-normal subgroup J is called F-**perfect**, if $J = [F, J]$.

absolutely perfect sets and canonical subgroups

Let X be an absolutely perfect set of elements. A subset Y of X is called **closed**, if

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Remark

The 1-1 correspondence above is not contrived (artificial) because: If $Y \subseteq X$ *and* $Y' =^{\langle X \rangle} \langle Y \rangle \cap X$ *then* Y' *is closed and*

$$^{\langle X \rangle}\langle Y \rangle =^{\langle X \rangle} \langle Y' \rangle.$$

general canonical subgroups

We fix for the rest of the talk

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Definition-Theorem

Let H be an F-normal subgroup of G.

Define the **general canonical subgroup** *$can(H)$ of H by*

$$can(H) = {}^{N_G(H)}\langle H \cap X \rangle.$$

Then $can(H)$ is F-normal and ${}^{N_G(H)}F$-perfect (in a canonical way) and

$$can(can(H)) = can(H).$$

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In the case of classical-like groups over arbitrary rings, the canonical subgroups are precisely the relative elementary subgroups. In the case of quasi-finite rings, all the relative elementary subgroups are normal in G and so all general canonical subgroups are canonical and therefore relative elementary subgroups.

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Lemma

A general canonical subgroup C is canonical $\Longleftrightarrow N_G(C)$ normalizes $\langle C \cap X \rangle$.

canonical subgroups and bouquets

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$$bouq(B) = \{S(F, K) \mid can(K) = B\}$$

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For $B \in \mathfrak{Can}(G)$, define the **bouquet of sandwiches**

$$bouq(B) = \{S(F, K) \mid can(K) = B\}$$

and the **bouquet of groups**

$$Bouq(B) = \{subgroups\ H \subseteq G \mid H \in S(F, K) \in bouq(B)\}.$$

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Let $B \in \mathfrak{Can}(G)$. Define the **cloud set of B** or simply **cloud of B**

$$cl(B) = \left\{ z \in G \mid can[F, \langle z \rangle] \subseteq B \right\}.$$

clouds and the cloud theorem

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Define the **cloud groups of B** or simply **group cloud of B**

$$Cl(B) = \{H \mid H \in S(F, K), can(K) \subseteq B\}.$$

Cloud Theorem

Let $B \in \mathfrak{Can}(G)$. Then each group of $Cl(B)$ is contained in $cl(B)$. Furthermore, these are the only F-normal subgroups contained in $cl(B)$ and they cover $cl(B)$.

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Finally, if there is a subgroup $H \in Cl(B)$ *with the property that if* $z \notin H$ *then* $[F, \langle z \rangle] \not\subseteq B$, *then* $cl(B)$ *is a group (i.e. is closed under multiplication) and* $H = cl(B)$.

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In the case of classical-like groups over quasi-finite rings, the clouds correspond to the elements of the full congruence subgroups.

Corollary of the cloud theorem

Let R be a ring. Call the sandwich classification of $GL_n(R)$ **standard**, if it is the same as that when R is quasi-finite.

Corollary

The sandwich classification of $GL_n(R)$ is standard $\Longleftrightarrow$ $E_n(R)$ is normal in $GL_n(R)$ and for every ideal I if R the cloud of $cl(1)$ of $GL_n(R/I)$ is the center$(GL_n(R/I))$.

Thank you for your attention