Groupoid models and C*-algebras of diagrams of groupoid correspondences

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Motivation and Summary

- Many C*-algebras may also be described as the groupoid C*-algebra of an étale, locally compact groupoid:
  - crossed products $C_0(X) \rtimes G$
  - (topological, higher-rank) graph C*-algebras
  - Nekrashevych's algebras of self-similar groups
  - simple Kirchberg algebras
  - all classifiable simple C*-algebras???
- Some of these C*-algebras are already built from "dynamical, topological" data.
- We are going to
  - encode this data through groupoid correspondences or, more generally, diagrams of groupoid correspondences
  - from this, build a C*-correspondence or, more generally, a product system
  - define a groupoid model from that
  - compare the groupoid C*-algebra of the groupoid model to the Cuntz–Pimsner algebra of the C*-correspondence.

Étale (locally compact) groupoids – the definition

Definition (data of an étale groupoid)

$\mathcal{G}^{(0)}$	objects, units	locally compact, Hausdorff space
$\mathcal{G}$	arrows	locally (quasi)compact space
r	range	local homeomorphism $\mathcal{G} \to \mathcal{G}^{(0)}$
s	source	local homeomorphism $\mathcal{G} \to \mathcal{G}^{(0)}$
$\mathcal{G}^{(2)}$	composables	$\{(g,h) \in \mathcal{G} \times \mathcal{G} : s(g) = r(h)\}$
m	multiplication	continuous map $\mathcal{G}^{(2)} \to \mathcal{G}$, $(g,h) \mapsto g \cdot h$

Definition (conditions for an étale groupoid)

- m is associative, $r(g \cdot h) = r(g)$, $s(g \cdot h) = s(h)$.
- m is unital. The unit map $\mathcal{G}^{(0)} \to \mathcal{G}$ is a homeomorphism onto an open subset.
- Each arrow has an inverse.
  The inversion $\mathcal{G} \to \mathcal{G}$ is a homeomorphism.

An example: transformation groupoid for a group action

The data

Γ (discrete) group

X locally compact space

α $\Gamma \times X \to X$, group action by homeomorphisms

The following is a Hausdorff étale groupoid:

The transformation groupoid: $x \xrightarrow{\gamma} \gamma \cdot x$

$\mathcal{G}^{(0)}$ $\mathcal{G}^{(0)} := X$

r $r(\gamma, x) := \gamma \cdot x$

s $s(\gamma, x) := x$

m $(\gamma, \eta \cdot x) \cdot (\eta, x) := (\gamma \cdot \eta, x)$

Inverse semigroups

Definition

An inverse semigroup is a semigroup S such that for each $s \in S$ there is a unique $t \in S$ with $sts = s$ and $tst = t$. Write $s^* := t$.

- Let X be a set.
- A partial map on X is a map $f\colon U \to V$ for $U, V \subseteq X$.
- Let $X^+ := X \sqcup \{\infty\}$.
  Extend f to a map $f^+\colon X^+ \to X^+$ by $f(x) := \infty$ if $x \notin U$.
- The ordinary composite $f^+ \circ g^+$ for partial maps f, g is $(f \circ g)^+$ for a partial map $f \circ g$, defined by $(f \circ g)(x) := f(g(x))$ if $g(x)$ and $f(g(x))$ are defined, and undefined otherwise.

Theorem

Partial bijections $X \to X$ form an inverse semigroup.
Here f^* for a bijection $f\colon U \to V$ is $f^{-1}\colon V \to U$.
Any inverse semigroup embeds in partial bijections.

More structure on inverse semigroups

Let S be an inverse semigroup.

unit $1 \in S$ with $1 \cdot s = s = s \cdot 1$

zero $0 \in S$ with $0 \cdot s = 0 = s \cdot 0$

idempotent $e \in S$ with $e^2 = e$, $E(S) := \{\text{idempotents}\}$

Definition (partial order)

$s \leq t \iff s = ts^*s \iff s = ss^*t \iff$
$s = te$ for some $e \in E(S) \iff s = et$ for some $e \in E(S)$.

Example (inverse semigroup of partial bijections)

The identity on X is 1. The empty partial map is 0.
The idempotents are the identity maps of subsets.
$s \leq t \iff s$ is a restriction of t to a subset of its domain.

Theorem

The idempotents in S form a commutative subsemigroup.

Inverse semigroup actions

Let X be a locally compact space.

Definition (partial homeomorphism)

homeomorphism $\vartheta_s \colon U \xrightarrow{\sim} V$ for open subsets $U, V \subseteq X$

Definition (action of an inverse semigroup S on X)

homomorphism $S \to \{$partial homeomorphisms of $X\}$
It maps $s \in S$ to a homeomorphism $\vartheta_s \colon D_{s^*s} \xrightarrow{\sim} D_{ss^*}$
for open subsets $D_e \subseteq X$ for $e \in E(S)$.

Remark

If S has no unit, we adjoin one to it and extend the action by
$1 \cdot x := x$ for all $x \in X$.

Transformation groupoid of an inverse semigroup action

$\mathcal{G}^{(0)}$	X
$\tilde{\mathcal{G}}$	$\bigsqcup_{t \in S} D_{t^*t}$, elements (t, x) with $x \in D_{t^*t}$
s	$s(t, x) := x$
r	$r(t, x) := \vartheta_t(x)$
m	$(u, \vartheta_t(x)) \cdot (t, x) := (u \cdot t, x)$

Problem

$\tilde{\mathcal{G}}$ is not a groupoid.

Definition (equivalence relation $\sim$)

$(t, x) \sim (u, x) \iff \exists\, e \in E(S)$ with $x \in D_e$ and $t \cdot e = u \cdot e$.

Theorem

There is an étale groupoid $\mathcal{G} = X \rtimes_\vartheta S$ with $\mathcal{G} := \tilde{\mathcal{G}}/\!\sim$ with the quotient topology and r, s, m induced by the maps on $\tilde{\mathcal{G}}$.

It may happen that $\mathcal{G}$ is not Hausdorff.

Bisections of an étale groupoid

Definition (bisection)

open subset $U \subseteq \mathcal{G}$ such that $r|_U$, $s|_U$ are injective
It defines a partial homeomorphism of $\mathcal{G}^{(0)}$:

$$\vartheta_U := r|_U \circ (s|_U)^{-1} \colon s(U) \to r(U)$$

- The bisections in $\mathcal{G}$ with the multiplication of subsets form an inverse semigroup $\mathrm{Bis}(\mathcal{G})$.
- ϑ is an action of $\mathrm{Bis}(\mathcal{G})$ on $\mathcal{G}^{(0)}$.

Proposition (étale groupoids $\simeq$ inverse semigroup actions)

The transformation groupoid of the $\mathrm{Bis}(\mathcal{G})$*-action on* $\mathcal{G}^{(0)}$ *is again* $\mathcal{G}$*.*

Warning

Let ϑ be an action of an inverse semigroup S on X.
There is a canonical homomorphism $S \to \mathrm{Bis}(X \rtimes_\vartheta S)$.
It need not be an isomorphism.

The C*-algebra of an étale groupoid

- Let U, V be bisections, $f_1 \in \mathrm{C_c}(U)$, $f_2 \in \mathrm{C_c}(V)$. Extend f_1, f_2 by 0 outside U and V.
- $(f_1 * f_2)(g) := \sum_{r(x)=r(g)} f_1(x) f_2(x^{-1}g)$
- $f^*(g) := \overline{f(g^{-1})}$
- Then $f_1 * f_2 \in \mathrm{C_c}(U \cdot V)$, $f^* \in \mathrm{C_c}(U^*)$.
- Let $\mathfrak{S}(\mathcal{G})$ be the linear span of $\mathrm{C_c}(U)$ for all bisections of $\mathcal{G}$.
- $\mathcal{G}$ is Hausdorff $\iff$ $\mathfrak{S}(\mathcal{G}) = \mathrm{C_c}(\mathcal{G})$
- The formulas above make $\mathfrak{S}(\mathcal{G})$ a *-algebra.

Definition (groupoid C*-algebra)

$\mathrm{C}^*(\mathcal{G})$ is the completion of $\mathfrak{S}(\mathcal{G})$ in its largest C*-seminorm.

Irreversible dynamics and topological correspondences

- Groups act on spaces by homeomorphisms.
  This describes reversible dynamics $\cong$ deterministic systems.

Question

How to encode irreversible dynamics?

- Basic case: non-invertible local homeomorphism $f\colon X \to X$.
- We are going to attach two étale groupoids to f.
- We combine both cases through topological correspondences.
- We interpret these groupoids in dynamical terms.

Lifting a continuous map to a homeomorphism

- Let $\sigma\colon X \to X$ be a continuous map.
- Take the projective limit $Y := \varprojlim (X, \sigma)$.
  σ induces shift homeomorphisms

$$\begin{aligned} \tilde{\sigma}\colon Y &\to Y, & \tilde{\sigma}^{-1}\colon Y &\to Y, \\ (x_0, x_1, \dots) &\mapsto (\sigma(x_0), x_0, \dots), & (x_0, x_1, \dots) &\mapsto (x_1, x_2, \dots). \end{aligned}$$

- Note $(\sigma(x_0), x_0, x_1, x_2, \dots) = (\sigma(x_0), \sigma(x_1), \sigma(x_2), \sigma(x_3), \dots)$.
- Take the transformation groupoid $Y \rtimes \mathbb{Z}$.
- If X is locally compact and σ is proper,
  then $Y \rtimes \mathbb{Z}$ is an étale, locally compact groupoid.

Theorem

Let X be a locally compact space and $\sigma\colon X \to X$ a proper, continuous map. Then $\mathrm{C}^*(Y \rtimes \mathbb{Z})$ is the Stacey crossed product of the endomorphism σ^* of $\mathrm{C}_0(X)$.

Physical interpretation

Question

Can $Y \rtimes \mathbb{Z}$ describe non-deterministic dynamical systems? – No.
It describes a deterministic system with incomplete records.

- Interpret X as the state space of a dynamical system.
- A point in Y is a family of states $x_n \in X$ with $\sigma(x_{n+1}) = x_n$ for all $n \in \mathbb{N}$.
- Interpret σ as a forward time step.
- A point in Y is a possible complete past history of the system.
- $\tilde{\sigma}(x_0, x_1, x_2) := (\sigma(x_0), x_0, x_1, \dots)$ and $\tilde{\sigma}^{-1}(x_0, x_1, x_2) := (x_1, x_2, x_3, \dots)$ are the forward and backward time steps on Y.

The groupoid of Renault and Deaconu

X locally compact space

σ local homeomorphism $X \to X$

- The following is a groupoid:

  $\mathcal{G}^{(0)}$ X

  $\mathcal{G}$ $\{(x, n, y) : \exists k, l \geq 0 : l - k = n,\ \sigma^k(x) = \sigma^l(y)\}$

  r $r(x, n, y) := x$

  s $s(x, n, y) := y$

  m $(x, n, y) \cdot (y, m, z) := (x, n + m, z)$

- We describe a topology on $\mathcal{G}$ to make it an étale groupoid. We give a basis of the topology consisting of open bisections.
- Let $k, l \in \mathbb{N}$, $U_1, U_2 \subseteq X$ be open subsets such that $\sigma^k|_{U_1}$ and $\sigma^l|_{U_2}$ are injective and $\sigma^k(U_1) = \sigma^l(U_2)$. The set of $(x, l - k, y)$ with $x \in U_1$, $y \in U_2$, $\sigma^k(x) = \sigma^l(y)$ is a bisection of $\mathcal{G}$.
- These subsets are a basis for a topology on $\mathcal{G}$.

Physical interpretation

- The object space of $\mathcal{G}$ is the state space X.
- σ describes the backwards time step.
- For $k \in \mathbb{N}$, let $x \sim_k y \iff \sigma^k(x) = \sigma^k(y)$.
- This is an increasing sequence of equivalence relations on X.
- Each $\sim_k$ is closed in $X \times X$ and gives a proper étale groupoid without isotropy: "proper equivalence relation".
- The union $\sim_\infty = \bigcup_k \sim_k$ is the set of all $(x, y) \in X^2$ with $(x, 0, y) \in \mathcal{G}$.
- $x \sim_\infty y \iff x, y$ have a common past.
- Let $n \in \mathbb{N}$. Then
  $(x, n, y) \in \mathcal{G} \iff x, \sigma^n(y)$ have a common past,
  $(x, -n, y) \in \mathcal{G} \iff \sigma^n(x), y$ have a common past.

Topological correspondences

- We have built two étale groupoids for a local homeomorphism $\sigma\colon X \to X$.
- One groupoid treats σ as the forward time step of a system where the present does not determine the past.
- One groupoid treats σ as the backward time step of a system where the present does not determine the future.
- A topological correspondence describes dynamical systems where the present determines neither future nor past.
- To build an étale groupoid from a topological correspondence, we first build a space of complete histories.
  It carries a dynamical system
  where the present determines the past.
  Then we take the Renault–Deaconu groupoid.

Topological correspondences – the definition

Definition

A topological correspondence on a locally compact space X is a locally compact space Z with a local homeomorphism $\sigma\colon Z \to X$ and a continuous map $\varrho\colon Z \to X$.
It is called proper if ϱ is a proper map.

Interpretation

X	possible situations
Z	possible events during one time step
$\varrho(z)$	situation after event z
$\sigma(z)$	situation before event z

Composition of topological correspondences

Lemma

Let $(Z_i, \varrho_i, \sigma_i)$ for $i = 1, 2$ be topological correspondences on X.

$$Z_1 \circ Z_2 := \{(z_1, z_2) \in Z_1 \times Z_2 : \sigma_1(z_1) = \varrho_2(z_2)\}$$

with $\varrho(z_1, z_2) := \varrho_1(z_1)$, $\sigma(z_1, z_2) := \sigma_2(z_2)$
is a topological correspondence on X.
It is proper if both factors are proper topological correspondences.

Interpretation

$Z^{\circ n}$ is the set of chains of events that can occur in n time steps.
ϱ and σ are the situations after and before such a chain of n events.

Complete histories

Let (Z, ϱ, σ) be a topological correspondence on X.

Definition (space of complete histories)

Let $\Omega = \varprojlim (Z^{\circ n}, \pi_n)$ for the maps
$\pi_n \colon Z^{\circ n} \to Z^{\circ n-1}$, $(z_0, \dots, z_n) \mapsto (z_0, \dots, z_{n-1})$.
An element of Ω is $(z_j)_{j \in \mathbb{N}} \in \prod_{\mathbb{N}} Z$ with $\sigma(z_j) = \varrho(z_{j+1})$ for $j \in \mathbb{N}$.
Define $\Sigma \colon \Omega \to \Omega$, $(z_0, z_1, z_2, \dots) \mapsto (z_1, z_2, z_3, \dots)$.

Lemma

Σ is a local homeomorphism on Ω.
If ϱ is proper, then Ω is locally compact.

Proof.

- $\Omega \cong Z \times_{\sigma, X, \varrho} \Omega$
- Σ becomes the canonical projection $Z \times_{\sigma, X, \varrho} \Omega \to \Omega$.
- This is a local homeomorphism because $\sigma \colon Z \to X$ is.
- Apply Tychonov's Theorem to the preimage in Ω of a compact subset of X. □

The groupoid defined by a topological correspondence

- Let (Z, ϱ, σ) be a topological correspondence on X.
- We have built the space Ω of complete histories and a local homeomorphism $\Sigma\colon \Omega \to \Omega$.
- The map Σ forgets the last event in a complete history.
- We interpret it as a backward time step in a dynamical system where the present determines the past but not the future.
- We associate to this the Renault–Deaconu groupoid of $\Sigma\colon \Omega \to \Omega$.

An equivalent language

top. correspondence	directed topological graph
X	vertex space
Z	edge space
$\varrho(z), \sigma(z)$	initial and terminal vertex of edge z
Ω	space of infinite paths
$\mathcal{G}(\Sigma)$	groupoid model of top. graph C*-algebra
	if ϱ is proper and surjective

An easy example

X	$X = \mathrm{pt}$
Z	$Z = \{0, 1\}$
Ω	consists of infinite binary words $(z_n)_{n \in \mathbb{N}}$ with $z_n \in \{0, 1\}$. Remarkably complicated given X and Z.
Σ	shift on infinite binary words
$\sim_\infty$	tail equivalence: $z_n = w_n$ for $n \gg 0$

Proposition (Renault)

The groupoid C^**-algebra in this example is the Cuntz algebra* $\mathcal{O}_2$*.*

Étale groupoid correspondences

- We are going to define an analogue of topological correspondences between two groupoids.
- A (proper) groupoid correspondence on a space X is a (proper) topological correspondence on X.
- A proper groupoid correspondence on a group $\mathcal{G}$ is a self-similarity of $\mathcal{G}$.
- A proper groupoid correspondence on $V \rtimes \mathcal{G}$ for an action of a group $\mathcal{G}$ on a discrete set V is almost a self-similar graph with vertex set V (Exel–Pardo). We get the slightly more general self-similar groupoid actions on graphs by Laca–Raeburn–Ramagge–Whittaker.

Actions of étale groupoids

- Let $\mathcal{G}$ be an étale groupoid and Z a topological space.
- A (left) $\mathcal{G}$-action on Z consists of a continuous map $\varrho\colon Z \to \mathcal{G}^{(0)}$ ("anchor map") and a continuous map $\mu\colon \mathcal{G} \times_{s,\mathcal{G}^{(0)},\varrho} Z \to Z$, $(g, z) \mapsto g \cdot z$, such that
  - $1_{\varrho(z)} \cdot z = z$ for all $z \in Z$, and
  - $(g_1 \cdot g_2) \cdot z = g_1 \cdot (g_2 \cdot z)$ if $s(g_1) = r(g_2)$, $s(g_2) = \varrho(z)$.
- The action is free and proper if $\varphi\colon \mathcal{G} \times_{s,\mathcal{G}^{(0)},\varrho} Z \to Z \times Z$, $(g, z) \mapsto (g \cdot z, z)$, is a homeomorphism onto a closed subset.
- The image of φ is closed $\iff$ $\mathcal{G}\backslash Z$ is Hausdorff in the quotient topology.
- Right $\mathcal{G}$-actions are defined similarly. We write σ for their anchor maps.

Étale groupoid correspondences – the definition

Definition (étale groupoid correspondence)

Let $\mathcal{G}$ and $\mathcal{H}$ be étale groupoids.
An étale groupoid correspondence $\mathcal{G} \leftarrow \mathcal{H}$ is
a locally (quasi)compact space Z with commuting actions
$\mathcal{G} \curvearrowright Z \curvearrowleft \mathcal{H}$, such that

- $\sigma\colon Z \to \mathcal{H}^{(0)}$ is a local homeomorphism, and
- the right $\mathcal{H}$-action on Z is free and proper.

A groupoid correspondence is proper/tight
if the map $Z/\mathcal{H} \to \mathcal{G}^{(0)}$ induced by ϱ is proper/a homeomorphism.

Remark

We view a groupoid correspondence as an arrow from $\mathcal{H}$ to $\mathcal{G}$.
We go from right to left because we think of ϱ and σ as
generalised range and source maps.

Some variations

- The technicalities in the definition are essential.
- There is half a dozen different interesting classes of commuting actions for two groupoids.
- A locally (quasi)compact space Z with commuting actions $\mathcal{G} \curvearrowright Z \curvearrowleft \mathcal{H}$ is
  - a Morita equivalence if both actions are free and proper and ϱ and σ induce homeomorphisms $Z/\mathcal{H} \cong \mathcal{G}^{(0)}$ and $\mathcal{G}\backslash Z \cong \mathcal{H}^{(0)}$;
  - a Hilsum–Skandalis morphism $\mathcal{G} \to \mathcal{H}$ if the $\mathcal{H}$-action is free and proper and ϱ induces a homeomorphism $Z/\mathcal{H} \cong \mathcal{G}^{(0)}$.
- A groupoid correspondence is tight $\iff$ it is also a Hilsum–Skandalis morphism.
- Unlike a Hilsum–Skandalis morphism, a groupoid correspondence $\mathcal{G} \leftarrow \mathcal{H}$ induces a C^*-correspondence between $C^*(\mathcal{G})$ and $C^*(\mathcal{H})$.

Examples of groupoid correspondences

- A (proper) groupoid correspondence on a space X is a (proper) topological correspondence on X.
- A proper groupoid correspondence on a group G is a self-similarity of G.
- A proper groupoid correspondence on $V \rtimes G$ for an action of a group G on a discrete set V is almost a self-similar graph with vertex set V (Exel–Pardo). We get the slightly more general self-similar groupoid actions on graphs by Laca–Raeburn–Ramagge–Whittaker.

Topological correspondences as groupoid correspondences

- Let $\mathcal{G}$ and $\mathcal{H}$ be locally compact spaces viewed as groupoids with only identity arrows.
- A groupoid correspondence $\mathcal{G} \leftarrow \mathcal{H}$ is a locally compact space Z with a local homeomorphism $\sigma\colon Z \to \mathcal{H}$ and a continuous map $\varrho\colon Z \to \mathcal{G}$.
- The map $Z \times_{\sigma,\mathcal{H},\mathrm{id}} \mathcal{H} \to Z \times Z$, $(z,h) \mapsto (z,z)$, is always a homeomorphism onto its image. The image is closed $\iff$ Z is Hausdorff.
- The correspondence is proper $\iff$ ϱ is proper.
- The correspondence is tight $\iff$ ϱ is a homeomorphism.
- In the tight case, the groupoid correspondence is isomorphic to one where $Z = \mathcal{G}$ and $\varrho = \mathrm{id}_{\mathcal{G}}$.
- Thus a tight groupoid correpondence $\mathcal{G} \leftarrow \mathcal{H}$ is equivalent to a local homeomorphism $\sigma\colon \mathcal{G} \to \mathcal{H}$.

Self-similarities of groups

- Let G and H be groups and let Z be a groupoid correspondence $G \leftarrow H$.
- $\sigma\colon Z \to H^{(0)} = \{\mathrm{pt}\}$ local homeomorphism $\Longrightarrow$ Z discrete
- Since the right H-action on Z is free and proper, $Z \cong A \times H$ with $(a, h_1) \cdot h_2 = (a, h_1 \cdot h_2)$.
  Here A may be taken to be a fundamental domain or Z/H.
- Must have $g \cdot (a, h) = (g \cdot a, g|_a \cdot h)$ for some left G-action on A and some map $G \times A \to H$, $(g, a) \mapsto g|_a$.
- $g_1 \cdot (g_2 \cdot (a, h)) = (g_1 \cdot g_2) \cdot (a, h) \iff$ $(g_1 \cdot g_2)|_a = g_1|_{g_2 \cdot a} \cdot g_2|_a$.
- The groupoid correspondence is proper $\iff$ A is finite.
  If $G = H$, then this is a self-similarity of G.
  Nekrashevych calls Z a "covering permutational bimodule".
- The groupoid correspondence is tight $\iff$ $|A| = 1$ $\iff$ $Z \cong H$ as a right H-space.
  Then the left action is a group homomorphism $G \to H$.