Fine structure of C*-algebras associated to topological dynamics

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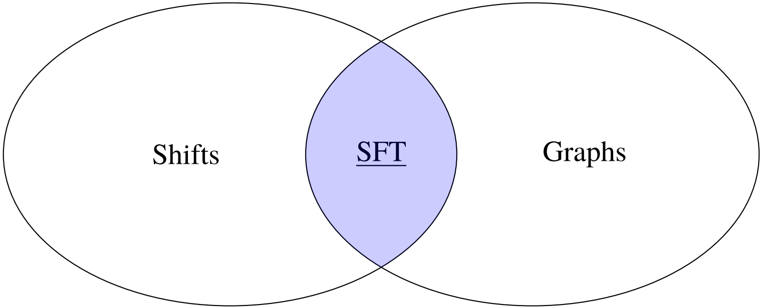
Overview

Classical results

Symbolic dynamical systems



Dynamics: Shifts of finite type

Let E_A be a finite directed graph with adjacency matrix A over $\mathbb{N}$. Let X_A and Λ_A be the right-infinite and bi-infinite paths in E_A with the shift operation $\sigma_A(x)_i = x_{i+1}$.

Definition

Λ_A and Λ_B are

- conjugate if there is a homeomorphism $h\colon \Lambda_A \longrightarrow \Lambda_B$ with $h \circ \sigma_A = \sigma_B \circ h$.
- flow equivalent if the suspensions $\Lambda_A \times \mathbb{R}/\sim$, where

$$(\sigma_A(x), t) \sim (x, t+1),$$

  are conjugate with respect to the natural $\mathbb{R}$-action.

Conjugacy is hard

Example

Are the systems given by the matrices

$$\begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 12 \\ 1 & 1 \end{bmatrix}$$

conjugate?

There is no known algorithm to decide this.

The flow problem

Conjugacy is hard! So…

Problem

Classify systems up to flow equivalence.

Theorem (Bowen, Franks)

For irreducible nonpermutation matrices A, the signed Bowen–Franks invariant

$$[\det(\mathrm{id} - A), \operatorname{coker}(\mathrm{id} - A)]$$

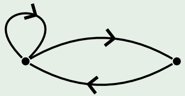
is complete for flow equivalence.

Note: $\operatorname{coker}(\mathrm{id} - A) \cong K_0(\mathcal{O}_A)$ — the *Cuntz–Krieger algebra*.

The flow problem

Example

The graphs



are flow equivalent; the signed BF-invariant is $[-1, 0]$. However, they are not conjugate (entropy).

Cuntz–Krieger algebras

Theorem (Cuntz–Krieger)

To every irreducible and nonpermutation $\mathbb{N}$-matrix A, there exists a universal unital C^*-algebra $\mathcal{O}_A$ generated by $|A|$ partial isometries with $K_0(\mathcal{O}_A) \cong \operatorname{coker}(\mathrm{id} - A)$. If Λ_A and Λ_B are flow equivalent, then

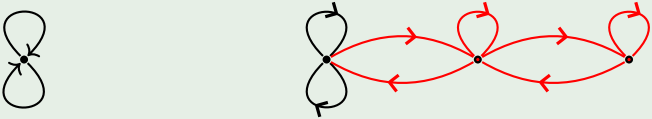
$$[\mathcal{O}_A \otimes \mathbb{K}, C(\mathsf{X}_A) \otimes c_0] \cong [\mathcal{O}_B \otimes \mathbb{K}, C(\mathsf{X}_B) \otimes c_0].$$

Here, $C(\mathsf{X}_A) \subset \mathcal{O}_A$ is the *diagonal* subalgebra.

The $\mathcal{O}_2$ — $\mathcal{O}_{2-}$ example

Example (The Cuntz-splice)

Consider the graphs



with matrices $A_2 = \begin{bmatrix} 2 \end{bmatrix}$ and $A_{2-} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Remark

$\mathcal{O}_2$ and $\mathcal{O}_{2-}$ are *-isomorphic.

Rigidity results

Theorem (Cuntz–Krieger, Matsumoto–Matui, CEOR)

Λ_A and Λ_B are flow equivalent if and only if

$$[\mathcal{O}_A \otimes \mathbb{K}, C(\mathsf{X}_A) \otimes c_0] \cong [\mathcal{O}_B \otimes \mathbb{K}, C(\mathsf{X}_B) \otimes c_0].$$

Method of proof: Groupoids and

Theorem (Matsumoto, Matsumoto–Matui, CEOR)

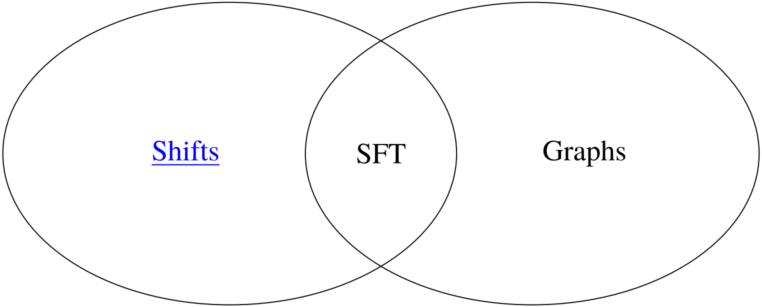
X_A and X_B are continuously orbit equivalent if and only if

$$[\mathcal{O}_A, C(\mathsf{X}_A)] \cong [\mathcal{O}_B, C(\mathsf{X}_B)]$$

and continuous orbit equivalence $\implies$ flow equivalence.

Overview

General shift spaces



General shift spaces

Definition (Morse–Hedlund)

A *general (one-sided) shift space* is a closed subspace $\mathsf{X} \subset \{1, \dots, N\}^{\mathbb{N}}$ which is shift-invariant ($\sigma(\mathsf{X}) \subset \mathsf{X}$). Similarly, there is a two-sided shift $\Lambda \subset \{1, \dots, N\}^{\mathbb{Z}}$ which is shift-invariant.

Problem

Are there structure results connecting general shift spaces with their C^*-algebras? (e.g. flow equivalence or conjugacy).

Note: there is a factor map $\rho \colon \Lambda \longrightarrow \mathsf{X}$ with $\rho((x)_{i \in \mathbb{Z}}) = (x)_{i \in \mathbb{N}}$, and the shift σ_{X} is local homeomorphism if and only if X is of *finite type*.

General shift spaces: Covers

The shift σ_{X} is only locally injective — $\sigma_{\mathsf{X}}(Z(\alpha))$ need not be open. We *force* them to be open!

Definition

Let $C_{\mathsf{X}}(\mu, \nu) := \{\nu x \in \mathsf{X} \mid \mu x \in \mathsf{X}\}$ and consider the commutative C^*-algebra

$$\mathcal{D}_{\mathsf{X}} := \overline{\operatorname{span}}\{1_{C_{\mathsf{X}}(\mu,\nu)} \mid \nu \text{ and } \mu \text{ are finite words}\}$$

inside the bounded (complex-valued) maps on X.

The cover: the spectrum $\tilde{\mathsf{X}} := \hat{\mathcal{D}}_{\mathsf{X}}$ has an induced action $\sigma_{\tilde{\mathsf{X}}}$ (a local homeomorphism), and a factor map $\pi_{\mathsf{X}} \colon \tilde{\mathsf{X}} \longrightarrow \mathsf{X}$.

$\mathcal{D}_{\mathsf{X}}$ is the ill-fated commutative algebra... which has evaded all attempts at analysis... Descriptions of $\hat{\mathcal{D}}_{\mathsf{X}}$ are somewhat terse and obscure...

Dokuchaev–Exel (2017)

Example: Shifts of finite type

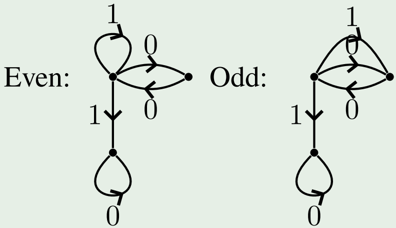
If $\mathsf{X} = \mathsf{X}_A$ is a shift of finite type, then $\pi_{\mathsf{X}} \colon \tilde{\mathsf{X}}_A \longrightarrow \mathsf{X}_A$ is a conjugacy!

Example: Sofic shifts

Definition (Weiss)

A *sofic shift* is a homomorphic image of a shift of finite type.

Example (The even and odd shifts)



The map defined by sending $1 \longmapsto 10$ defines a continuous orbit equivalence.

Example: Sturmian shifts

Example (Morse–Hedlund)

Let $\alpha \in (0,1) \setminus \mathbb{Q}$ and consider the rigid rotation $R_\alpha \colon [0,1) \longrightarrow [0,1)$ given by $R_\alpha(t) = t + \alpha \pmod 1$.
Consider $I_\alpha \colon [0,1) \longrightarrow \{0,1\}$ given by

$$I_\alpha(t) = \begin{cases} 0, & t \in [0, 1-\alpha), \\ 1, & t \in [1-\alpha, 1). \end{cases}$$

The *Sturmian shift* is then

$$\mathsf{X}_\alpha = \overline{\{(I_\alpha(R_\alpha^i(t)))_{i \in \mathbb{N}} : t \in [0,1)\}}.$$

Example: Sturmian shifts

Example (Morse–Hedlund)

The *Sturmian shift*

$$\mathsf{X}_\alpha = \overline{\{(I_\alpha(R^i_\alpha(t)))_{i\in\mathbb{N}} : t \in [0,1)\}}$$

is minimal with no periodic points and vanishing entropy. The cover $\tilde{\mathsf{X}}_\alpha$ can be identified with

$$\Lambda_\alpha \cup \{\text{countable set of isolated points}\}$$

Sturmian shifts are very rigid (continuous orbit equivalence = flow equivalence)!

Rigidity

Theorem (B–Carlsen)

Sofic shifts X *and* Y *(with technical condition) are continuously orbit equivalent if and only if*

$$[\mathcal{O}_{\mathsf{X}}, C(\mathsf{X})] \cong [\mathcal{O}_{\mathsf{Y}}, C(\mathsf{Y})].$$

Theorem (B–Carlsen)

Sofic shifts Λ_{X} *and* Λ_{Y} *(with technical condition) are flow equivalent if and only if*

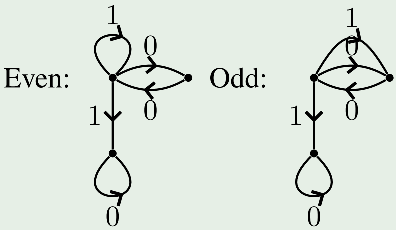
$$[\mathcal{O}_{\mathsf{X}} \otimes \mathbb{K}, C(\mathsf{X}) \otimes c_0] \cong [\mathcal{O}_{\mathsf{Y}} \otimes \mathbb{K}, C(\mathsf{Y}) \otimes c_0]$$

In particular, continuous orbit equivalence $\implies$ *flow equivalence.*

Example: Sofic shifts

Unfortunately, the even and odd shift do not satisfy the *technical condition*.

Example (The even and odd shifts)



The problem of classifying irreducible sofics is still open.

Rigidity

Theorem (B–Carlsen)

Λ_{X} and Λ_{Y} are flow equivalent if and only if there are

- *a *-isomorphism $\Phi\colon [\mathcal{O}_{\mathsf{X}} \otimes \mathbb{K}, C(\mathsf{X}) \otimes c_0] \longrightarrow [\mathcal{O}_{\mathsf{Y}} \otimes \mathbb{K}, C(\mathsf{Y}) \otimes c_0]$;*
- *a homomorphism $\psi^*\colon C(\mathsf{Y} \times \mathbb{N}, \mathbb{Z}) \longrightarrow C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$ such that*

$$\Phi \circ \beta_z^{\kappa_{\mathsf{X}\times\mathbb{N}}(\psi^*(\eta))} = \beta_z^{\kappa_{\mathsf{Y}\times\mathbb{N}}(\eta)} \circ \Phi,$$

  for $\eta \in C(\mathsf{Y} \times \mathbb{N}, \mathbb{Z})$ and $z \in \mathbb{T}$;
- *a homomorphism $\psi^\#\colon C(\mathsf{X} \times \mathbb{N}, \mathbb{Z}) \longrightarrow C(\mathsf{Y} \times \mathbb{N}, \mathbb{Z})$ such that*

$$\Phi \circ \beta_z^{\kappa_{\mathsf{X}\times\mathbb{N}}(\zeta)} = \beta_z^{\kappa_{\mathsf{Y}\times\mathbb{N}}(\psi^\#(\zeta))} \circ \Phi,$$

  for $\zeta \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$ and $z \in \mathbb{T}$; and
- *a positive isomorphism $H(\psi)\colon H^{\mathsf{Y}\times\mathbb{N}} \longrightarrow H^{\mathsf{X}\times\mathbb{N}}$ such that $H(\psi)([\eta]) = [\psi^*(\eta)]$ $\eta \in C(\mathsf{Y} \times \mathbb{N}, \mathbb{Z})$, and $H(\psi)^{-1}([\zeta]) = [\psi^\#(\zeta)]$ for $\zeta \in C(\mathsf{X} \times \mathbb{N}, \mathbb{Z})$.*

Rigidity

So . . . what about conjugacy?

Theorem (Carlsen–Rout)

Λ_A and Λ_B are conjugate if and only if

$$[\mathcal{O}_A \otimes \mathbb{K}, C(\mathsf{X}_A) \otimes c_0, \gamma^A \otimes \mathrm{id}] \cong [\mathcal{O}_B \otimes \mathbb{K}, C(\mathsf{X}_B) \otimes c_0, \gamma^B \otimes \mathrm{id}].$$

Here, $\gamma^A \colon \mathbb{T} \curvearrowright \mathcal{O}_A$ is a canonical circle action.

Theorem (B–Carlsen)

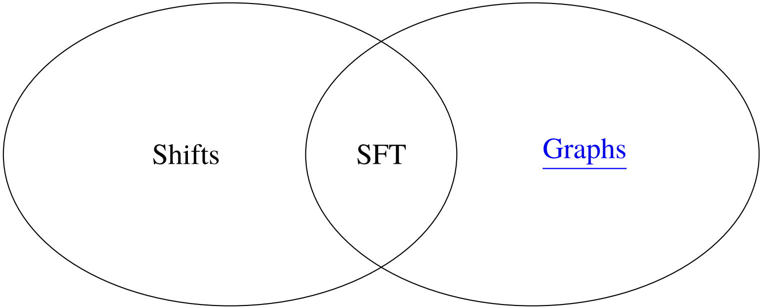
Λ_X and Λ_Y are conjugate if and only if

$$[\mathcal{O}_\mathsf{X} \otimes \mathbb{K}, C(\mathsf{X}) \otimes c_0, \gamma^\mathsf{X} \otimes \mathrm{id}] \cong [\mathcal{O}_\mathsf{Y} \otimes \mathbb{K}, C(\mathsf{Y}) \otimes c_0, \gamma^\mathsf{Y} \otimes \mathrm{id}].$$

Overview

1. Prelude: the flow problem
2. General shift spaces
3. Postlude: Graph algebras

Topological dynamics: Graphs



Structure-preserving *-isomorphism

Eilers and Ruiz: conjectures about *moves on graphs* and structure-preserving *-isomorphisms of graph C^*-algebras.

Inspired by study of conjugacy and flow:

- Williams: splitting vertices (according to incoming or outgoing edges); and
- Parry–Sullivan: stretching edges (time delay).

We have seen that flow corresponds to $[\mathcal{O}_A \otimes \mathbb{K}, C(\mathsf{X}_A) \otimes c_0]$.

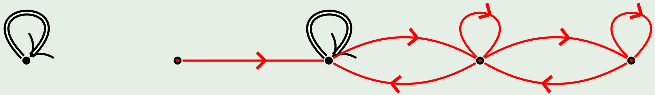
Problem 1: Infinite unital graphs

Does flow (splitting and stretching) correspond to stable diagonal-preserving *-isomorphism?

Problem 2: $\mathcal{O}_\infty$

Example (Unital Cuntz-splice)

Consider the graphs



with matrices $\begin{bmatrix}\infty\end{bmatrix}$ and $\begin{bmatrix}\infty & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0\end{bmatrix}$.

Are the C^*-algebras *-isomorphic in a way that preserves the diagonal? (think continuous orbit equivalence)

Problem 3: Shift equivalence

Definition

Λ_A and Λ_B are shift equivalent if their higher powers are conjugate.

For irreducible matrices:

$$\text{Conjugacy} \implies \text{Shift equivalence} \implies \text{Flow equivalence.}$$

Theorem (Bratteli–Kishimoto)

A pair of primitive graphs E and F are shift equivalent if and only if

$$[\mathcal{O}_A \otimes \mathbb{K}, \gamma^A \otimes \mathrm{id}] \cong [\mathcal{O}_B \otimes \mathbb{K}, \gamma^B \otimes \mathrm{id}].$$

Shift equivalence

Corollary

For primitive matrices

$$[\mathcal{O}_A \otimes \mathbb{K}, \gamma^A \otimes \mathrm{id}] \cong [\mathcal{O}_B \otimes \mathbb{K}, \gamma^B \otimes \mathrm{id}]$$

implies

$$[\mathcal{O}_A \otimes \mathbb{K}, C(\mathsf{X}_A) \otimes c_0] \cong [\mathcal{O}_B \otimes \mathbb{K}, C(\mathsf{X}_B) \otimes c_0].$$

(From shift to flow) Note: We need to change the *-isomorphism.