Non-commutative Stone dualities

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DISCLAIMER

This talk will concentrate on IDEAS rather than TECHNOLOGY.

1. Origins

All of our work is derived from the theory of *pseudogroups of transformations.* We shall reverse history and define such pseudogroups in terms of inverse semigroups.

A semigroup S is said to be *inverse* if for each $a \in S$ there exists a unique element a^{-1} such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$.

Two key immediate examples:

1. Groups are the inverse semigroups with exactly one idempotent.

2. Meet semilattices are the inverse semigroups in which every element is idempotent.

Inverse semigroups come equipped with an internally defined order.

Let S be an inverse semigroup. Define $a \leq b$ if $a = ba^{-1}a$.

Proposition *The relation* $\leq$ *is a partial order. In addition, if* $a \leq b$ *then* $a^{-1} \leq b^{-1}$ *and if also* $c \leq d$ *then* $ac \leq bd$.

This order is called the *natural partial order*.

Let S be an inverse semigroup. Elements of the form $a^{-1}a$ and aa^{-1} are idempotents. Denote by $\mathsf{E}(S)$ the set of idempotents of S.

Remarks

1. $\mathsf{E}(S)$ is a commutative subsemigroup or *semilattice*.

2. $\mathsf{E}(S)$ is an order ideal of S.

Observation Suppose that $a, b \leq c$. Then $ab^{-1} \leq cc^{-1}$ and $a^{-1}b \leq c^{-1}c$. Thus a necessary condition for a and b to have an upper bound is that $a^{-1}b$ and ab^{-1} be idempotent.

Define $a \sim b$ if $a^{-1}b$ and ab^{-1} are idempotent. This is the *compatibility relation*.

A subset is said to be *compatible* if each pair of distinct elements in the set are compatible.

Elements in inverse semigroups need to be compatible before they are even eligible to have a join.

- An inverse semigroup is said to have *finite (resp. infinite) joins* if each finite (resp. arbitrary) compatible subset has a join.

- An inverse semigroup is said to be *distributive* if it has finite joins and multiplication distributes over such joins.

- An inverse monoid is said to be a *pseudogroup* if it has infinite joins and multiplication distributes over such joins.

- An inverse semigroup is a *meet semigroup* if has has all binary meets.

A *pseudogroup of transformations* is a pseudogroup of partial bijective functions on a set.

The key example is the pseudogroup of all homeomorphisms between the open subsets of a topological space.

Such pseudogroups played an important rôle in the work of Charles Ehresmann.

If the topology on the set X is discrete we get the *symmetric inverse monoids* $I(X)$.

A *frame* is a complete distributive lattice in which finite meets distribute over infinite joins.

The open subsets of a topological space form a frame.

IDEA: Think of pseudogroups as non-commutative frames.

This idea motivates all our work and underpins this talk.

This idea did not arise in a vacuum:

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Commutative	**Non-commutative**
Frame	Pseudogroup
Distributive lattice	Distributive inverse semigroup
Boolean algebra	Boolean inverse semigroup
	Boolean inverse meet semigroup

Algebra	**Topology**
Semigroup	Locally compact
Monoid	Compact
Meet-semigroup	Hausdorff

In this talk, I will concentrate on *Boolean inverse monoids*.

“The fox knows many things, but the hedgehog knows one big thing.” — Archilochus

There is one idea driving this research:

think of inverse semigroups as non-commutative meet semilattices.

Technical point.

In the more general setting, one sets up an adjunction between a suitable category of pseudogroups and a suitable category of étale groupoids.

From this adjunction, categorical dualities can then be derived linking distributive inverse semigroups with what we term *spectral groupoids* and Boolean inverse semigroups with locally compact groupoids.

To do this, one needs a suitable notion of *coherence* for pseudogroups.

2. Boolean inverse semigroups

A distributive inverse semigroup is said to be *Boolean* if its semilattice of idempotents forms a (generalized) Boolean algebra.

Symmetric inverse monoids are Boolean.

Theorem [Paterson, Wehrung] *Let S be a subsemigroup of a ring with involution R such that S is an inverse semigroup with respect to the involution. Then there is a Boolean inverse semigroup T such that $S \subseteq T \subseteq R$.*

The above result is significant when viewing inverse semigroups in relation to C^*-algebras.

Theorem [Lawson] *Every inverse semigroup can be embedded in a universal Boolean inverse semigroup.*

We view categories as 1-sorted structures: everything is an arrow. Objects are identified with identity arrows.

A *groupoid* is a category in which every arrow is invertible.

We regard groupoids as 'groups with many identities'.

If G is a groupoid denote its set of identities by G_o.

A subset $A \subseteq G$ is called a *local bisection* if $A^{-1}A, AA^{-1} \subseteq G_o$.

Proposition *The set of all local bisections of a groupoid forms a Boolean inverse meet monoid.*

An inverse semigroup is *fundamental* if the only elements that centralize all idempotents are themselves idempotents.

Closely related to aperiodicity in higher-rank graphs.

A *closed ideal* in a Boolean inverse semigroup is an ideal closed under finite compatible joins.

A Boolean inverse semigroup is 0-*simplifying* if it contains no non-trivial closed ideals.

A Boolean inverse semigroup is *simple* if it is both fundamental and 0-simplifying.

Theorem [Lawson, Malandro]

1. *The finite Boolean inverse monoids are isomorphic to the inverse monoids of local bisections of finite discrete groupoids.* *Compare with the structure theory of finite Boolean algebras: finite sets replaced by finite groupoids.*

2. *The finite fundamental Boolean inverse monoids are precisely the finite direct products of finite symmetric inverse monoids.*

3. *The finite simple Boolean inverse monoids are precisely the finite symmetric inverse monoids.*

Examples: AF monoids

There is an analogy between finite symmetric inverse monoids I_n of all partial bijections on a finite set with n elements and the C^*-algebras $M_n(\mathbb{C})$.

Accordingly, define a Boolean inverse monoid to be *approximately finite* or *AF* if it is a direct limit of finite direct products of finite symmetric inverse monoids.

AF inverse monoids are fundamental Boolean inverse meet monoids.

3. Non-commutative Stone duality

A topological groupoid is said to be *étale* if its domain and range maps are local homeomorphisms.

Why étale? This is explained by the following result.

Theorem [Resende] *A topological groupoid is étale if and only if its set of open subsets forms a monoid under multiplication of subsets with the identity of the monoid being the space of identities.*

Etale groupoids therefore have a strong algebraic character.

A *Boolean space* is a compact Hausdorff space with a basis of clopen subsets.

A *Boolean groupoid* is an étale topological groupoid whose space of identities is a Boolean space.

If G is a Boolean groupoid denote by $\mathsf{KB}(G)$ the set of all compact-open local bisections.

A subset $A \subseteq S$ of a Boolean inverse monoid is called a *filter* if $a, b \in A$ implies that there is a $c \in A$ such that $c \leq a, b$, and if $a \in A$ and $a \leq b$ then $b \in A$. It is said to be *proper* if $0 \notin A$. A subset $A \subseteq S$ of a Boolean inverse monoid is called an *ultrafilter* if it is a maximal proper filter.

If S is a Boolean inverse monoid denote by $\mathsf{G}(S)$ the set of ultrafilters of S.

Technical point.

Ultrafilters in a Boolean inverse monoid behave much like cosets in a group.

If A is an ultrafilter then $\mathbf{d}(A) = (A^{-1}A)^{\uparrow}$, the elements above those in $A^{-1}A$, is also an ultrafilter and an inverse subsemigroup. If A is an ultrafilter then $\mathbf{r}(A) = (AA^{-1})^{\uparrow}$ is also an ultrafilter and an inverse subsemigroup.

Let $a \in A$. Then

$$A = (a\mathbf{d}(A))^{\uparrow}.$$

If A and B are ultrafilters define $A \cdot B = (AB)^{\uparrow}$ only when $\mathbf{d}(A) = \mathbf{r}(B)$. This provides us with a groupoid multiplication on the set of ultrafilters.

Theorem [Non-commutative Stone duality I, Lawson & Lenz]

1. *If S is a Boolean inverse monoid then* $\mathsf{G}(S)$ *is a Boolean groupoid, called the* Stone groupoid *of S.*

2. *If G is a Boolean groupoid then* $\mathsf{KB}(G)$ *is a Boolean inverse monoid.*

3. *If S is a Boolean inverse monoid then* $S \cong \mathsf{KB}(\mathsf{G}(S))$.

4. *If G is a Boolean groupoid then* $G \cong \mathsf{G}(\mathsf{KB}(G))$.

There are many special cases of the above result. Here, I shall mention just two.

Theorem [Non-commutative Stone duality II, Lawson & Lenz]

1. *The groupoid* $\mathsf{G}(S)$ *is Hausdorff if and only if* S *is a meet monoid.*

2. S *is a simple Boolean inverse monoid if and only if* $\mathsf{G}(S)$ *is effective and minimal.*

4. Applications: Thompson-Higman type groups

Let $A_n = \{a_1, \ldots, a_n\}$ be a finite alphabet with $n \geq 2$ elements. Denote the free monoid on A_n by A_n^*.

A *morphism* between right ideals of A_n^* is the analogue of a right module morphism.

The *polycyclic inverse monoid* P_n is the inverse monoid of all bijective morphisms between principal right ideals of A_n^* together with the empty partial function. This inverse monoid arises naturally in connection with pushdown automata and context-free languages.

The *polycyclic distributive inverse monoid* D_n is the inverse monoid of all bijective morphisms between the finitely generated right ideals of A_n^* together with the empty partial function.

Define $\equiv$ on D_n by $a \equiv b$ if and only if for all $0 < x \leq b$ we have that $a \wedge x \neq 0$ and for all $0 < y \leq a$ we have that $b \wedge y \neq 0$.

This definition is due to Lenz.

Theorem [Lawson]

1. *$C_n = D_n/\equiv$ is a Boolean inverse monoid, called the* Cuntz inverse monoid, *whose group of units is the Thompson group $G_{n,1}$.*

2. *The map $P_n \to C_n$ is universal to those Boolean inverse monoids which convert $a_1a_1^{-1}, \ldots, a_na_n^{-1}$ to a join equal to* 1*. This means that C_n is the* tight completion *of P_n.*

3. *The groupoid associated with the Boolean inverse monoid C_n is isomorphic to the set of triples $(xw, \mid x \mid - \mid y \mid, yw)$, where x and y are finite strings and w is a right-infinite string, with a groupoid product.*

The above theory generalizes to classes of higher-rank graphs (work with A. Vdovina and, more recently, with both A. Vdovina and A. Sims).

Non-commutative Stone duality computes the *correct* groupoids in these cases.

This was one of the motivations of the theory: why invent the wheel twice?

5. Applications: MV algebras

In lieu of a definition: MV algebras are to multiple-valued logic as Boolean algebras are to classical two-valued logic.

Denote by $S/\mathscr{J}$ the poset of principal ideals of S. If this is a lattice we say that S satisfies the *lattice condition*. The following is a semigroup version of a theorem of Mundici.

Theorem [Lawson-Scott] *Every countable MV algebra is isomorphic to the 'structure'* $S/\mathscr{J}$ *where* S *is AF and satisfies the lattice condition.*

Wehrung (2017) has generalized this result to *arbitrary* MV algebras.

Example The direct limit of $I_1 \to I_2 \to I_4 \to I_8 \to \ldots$ is the *CAR inverse monoid* whose associated MV algebra is that of the dyadic rationals in $[0, 1]$.

6. Research question

There is, up to isomorphism, exactly one countable, atomless Boolean algebra. We call it the *Tarski* algebra.

Under classical Stone duality, the Stone space of the Tarski algebra is the *Cantor space.*

We define a *Tarski monoid* to be a countable Boolean inverse meet monoid whose semilattice of idempotents is a Tarski algebra.

Problem: classify the simple Tarski monoids.

The groups of units of such groups are analogues of the Thompson-Higman groups.

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